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Lagrangian Grassmannian in infinite dimension

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ABSTRACT

Given a complex structure J on a real (finite or infinite dimensional) Hilbert space $\mathcal{H}$, we study the geometry of the Lagrangian Grassmannian $\Lambda(\mathcal{H})$ of $\mathcal{H}$, i.e. the set of closed linear subspaces $L \subset \mathcal{H}$ such that $J(L) = L^{\perp}$. The complex unitary group $U(\mathcal{H}_J)$, consisting of the elements of the orthogonal group of $\mathcal{H}$ which are complex linear for the given complex structure, acts transitively on $\Lambda(\mathcal{H})$ and induces a natural linear connection in $\Lambda(\mathcal{H})$. It is shown that any pair of Lagrangian subspaces can be joined by a geodesic of this connection. A Finsler metric can also be introduced, if one regards subspaces L as projections p_L (=the orthogonal projection onto L) or symmetries $\epsilon_L = 2p_L - I$, namely measuring tangent vectors with the operator norm. We show that for this metric the Hopf–Rinow theorem is valid in $\Lambda(\mathcal{H})$: a geodesic joining a pair of Lagrangian subspaces can be chosen to be of minimal length. A similar result holds for the unitary orbit of a Lagrangian subspace under the action of the k-Schatten unitary group ($2 \leq k \leq \infty$), with the Finsler metric given by the k-norm.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional real Hilbert space with a complex structure, that is, an isometric operator $J : \mathcal{H} \to \mathcal{H}$ with $J^* = -J$ and $J^2 = -I$. The (non degenerate) symplectic form is given by $w(\xi, \eta) = \langle J\xi, \eta \rangle$. As usual, one defines a complex Hilbert space, denoted $\mathcal{H}_J$, endowing $\mathcal{H}$ with the complex inner product $\langle \xi, \eta \rangle_J = \langle \xi, \eta \rangle - \mathrm{i}w(\xi, \eta)$. The complex structure J enables one to multiply vectors in $\mathcal{H}$ by complex numbers in the usual way: if $z = x + \mathrm{i}y \in \mathbb{C}$, and $\xi \in H$, then $z\xi = x\xi + yJ(\eta)$.

The purpose of this paper is the geometric study of the *Lagrangian Grassmannian* $\Lambda(\mathcal{H})$, the space of Lagrangian subpaces of $\mathcal{H}$, i.e. closed subspaces $L \subset \mathcal{H}$ such that

$$J(L) = L^{\perp}.$$

The Lagrangian Grassmannian $\Lambda(n)$ of $\mathcal{H} = \mathbb{R}^n \times \mathbb{R}^n$ ($J(x, y) = (-y, x)$) was introduced by V.I. Arnold in [2] in 1967. He showed the connection between the topology of $\Lambda(n) \simeq O(n)/U(n)$ and the index introduced by Maslov for closed curves on a Lagrangian manifold $M \subset \mathbb{R}^{2n}$. These notions have been generalized to infinite dimensional Hilbert spaces (see [9] and references therein), and have found several applications to Algebraic Topology, Differential Geometry and Physics.

In this paper we consider a natural linear connection in $\Lambda(\mathcal{H})$, and focus on the geodesic structure of this manifold. The Hopf–Rinow theorem states that any two points on a complete, finite dimensional Riemannian manifold can be joined by a

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minimal geodesic. It is well known [10] that it is no longer true in infinite dimensions. Two points may not be even joined by a geodesic [4]. The manifold $\Lambda(\mathcal{H})$ is not even Riemannian, the natural metric available for the tangent spaces, as it will be clear, is the usual norm of operators in the Hilbert space $\mathcal{H}$. Our main result here shows that any two Lagrangian subspaces $L_0, L_1 \subset \mathcal{H}$ can be joined by a minimal geodesic. Denote by p_L the orthogonal projection onto L. In general two projections p_{L_0}, p_{L_1} verify $\|p_{L_0} - p_{L_1}\| \leq 1$. We show that if $\|p_{L_0} - p_{L_1}\| = 1$ there can be two or infinite many minimal geodesics of $\Lambda(\mathcal{H})$ joining L_0 and L_1. If $\|p_{L_0} - p_{L_1}\| < 1$, the minimal geodesic is unique.

The *real Grassmannian*, the space of closed subspaces of $\mathcal{H}$, will be denoted $Gr(\mathcal{H})$. The space of closed complex subspaces, i.e. subspaces $S \subset \mathcal{H}$ such that $z\xi \in S$ whenever $z \in \mathbb{C}$ and $\xi \in S$, or equivalently

$$J(L) = L,$$

will be denoted the *complex Grassmannian* $Gr(\mathcal{H}_J)$.

Clearly $Gr(\mathcal{H}_J) \subset Gr(\mathcal{H})$ and $\Lambda_J(\mathcal{H}) \subset Gr(\mathcal{H})$. It is known (see [7,9]) that the three sets are differentiable manifolds, and that the inclusions are submanifolds. Also it is clear that $Gr(\mathcal{H}_J) \cap \Lambda_J(\mathcal{H}) = \emptyset$.

If $S \in Gr(\mathcal{H})$, then clearly $S \in Gr(\mathcal{H}_J)$ if and only if $p_S J = J p_S$. Also it is clear in this case that p_S is the $\langle\ ,\ \rangle_J$-orthogonal projection onto S. More generally, if $\mathcal{B}(\mathcal{H})$ denotes the space of (real) linear operators in $\mathcal{H}$, the space $\mathcal{B}(\mathcal{H}_J)$ of complex linear operators consists of all elements $a \in \mathcal{B}(\mathcal{H})$ such that $aJ = Ja$. Moreover, $L \in Gr(\mathcal{H})$ is a Lagrangian subspace if and only if $p_L J + J p_L = J$.

It is customary to parametrize closed subspaces via orthogonal projections, $S \leftrightarrow p_S$, in order to carry on geometric or analytic computations. We shall also consider here, as in [14], an alternative description using symmetries. Denote by $\epsilon_S = 2p_S - I$, i.e. the symmetric orthogonal transformation which acts as the identity in S and minus the identity in $S^\perp$. Therefore we identify

$$S \leftrightarrow p_S \leftrightarrow \epsilon_S,$$

and shall favor which suits best in each situation. With these identifications, one has that

$$Gr(\mathcal{H}_J) = \{\epsilon = \epsilon_S : \epsilon^* = \epsilon = \epsilon^{-1} \text{ and } \epsilon J = J\epsilon\},$$

and

$$\Lambda_J(\mathcal{H}) = \{\epsilon = \epsilon_L : \epsilon^* = \epsilon = \epsilon^{-1} \text{ and } \epsilon J = -J\epsilon\}.$$

That is complex subspaces commute with J whereas Lagrangian subspaces anti-commute with J.

Denote by $O(\mathcal{H})$ the orthogonal group of $\mathcal{H}$, i.e.

$$O(\mathcal{H}) = \{g \in \mathcal{B}(\mathcal{H}) : g^* = g^{-1}\}.$$

Here g^* denotes the transpose of g, we shall use the same notation for the adjoint of complex linear operators, and no confusion should arise. Clearly $O(\mathcal{H})$ acts on $Gr(\mathcal{H})$ by means of $g \cdot S = g(S)$, or equivalently, $g \cdot \epsilon_S = g\epsilon_S g^*$. Denote by $U(\mathcal{H}_J) \subset O(\mathcal{H})$ the subgroup

$$U(\mathcal{H}_J) = \{u \in O(\mathcal{H}) : uJ = Ju\},$$

i.e. the unitary group of the complex Hilbert space $\mathcal{H}_J$. This group $U(\mathcal{H}_J)$ acts both on $Gr(\mathcal{H}_J)$ and $\Lambda_J(\mathcal{H})$ in the same fashion. The actions of $O(\mathcal{H})$ on $Gr(\mathcal{H})$ and of $U(\mathcal{H}_J)$ on $Gr(\mathcal{H}_J)$ are locally transitive: if ϵ_1 and ϵ_2 are symmetries in the same Grassmannian, such that $\|\epsilon_1 - \epsilon_2\| < 2$, then they are conjugate by an element of the group. Therefore the orbits of the corresponding actions coincide with the connected components. For the case of the real and complex Grassmannians, the components are parametrized by the dimensions of the subspaces and their complements. For the Lagrangian Grassmannian, the action of $U(\mathcal{H}_J)$ is transitive, and in particular $\Lambda_J(\mathcal{H})$ is connected.

In this paper we introduce a linear connection (Section 2) and a Finsler metric (Section 3) in $\Lambda(\mathcal{H})$. The linear connection is the one defined in [14,7] for the whole Grassmannian $Gr(\mathcal{H})$, which restricts to $\Lambda(\mathcal{H})$: if Y is a field and X is a vector both tangent to $\Lambda(\mathcal{H})$, then the derivative $\nabla_X Y$ performed in $Gr(\mathcal{H})$ remains tangent to $\Lambda(\mathcal{H})$. The geodesics of $\Lambda(\mathcal{H})$ can therefore be computed. We show that the exponential map of $\Lambda(\mathcal{H})$ is onto (a property which in general $Gr(\mathcal{H})$ does not have), and thus any pair of Lagrangian subspaces can be joined by a geodesic curve. Moreover, we show that the geodesic can be chosen of minimal length for the Finsler metric given by the usual operator norm at each tangent space of $\Lambda(\mathcal{H})$. In other words, the Hopf–Rinow theorem is valid in $\Lambda(\mathcal{H})$ for this metric. We also consider the geometry of certain subsets of $\Lambda(\mathcal{H})$. In Section 4 we consider the graphs of unbounded self-adjoint Fredholm operators, which form an open subset of $\Lambda(\mathcal{H})$. In Section 5 we study the submanifolds obtained as orbits of the Fredholm unitary group $U_c(\mathcal{H}_J)$,

$$U_c(\mathcal{H}_J) = \{u \in U(\mathcal{H}_J) : u - I \text{ is compact}\}.$$

These are shown to be geodesically convex: if two Lagrangian subspaces lie in the same orbit, then the minimal geodesics which join them in $\Lambda(\mathcal{H})$ remain inside the orbit. With the same technique, we treat orbits of a subspace under the action of the k-Schatten unitary groups $U_k(\mathcal{H}_J)$,

$$U_k(\mathcal{H}_J) = \{u \in U(\mathcal{H}_J) : u - I \in B_k(\mathcal{H}_J)\},$$

where $B_k(\mathcal{H}_J)$ denotes the k-Schatten ideal of $\mathcal{H}_J$, for $k \geq 2$. Results analogous to the compact case are obtained, the Finsler metric considered here is the one induced by the k-norm at every tangent space.