Journal of Geometry and Physics 59 (2009) 354–368

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Quantum families of maps and quantum semigroups on finite quantum spaces

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ARTICLE INFO

Article history:
Received 30 August 2007
Accepted 19 November 2008
Available online 24 November 2008

MSC:
primary 58B32
46L85
secondary 46L89

Keywords:
Quantum space
Quantum family
Quantum semigroup
Quantum group

ABSTRACT

Quantum families of maps between quantum spaces are defined and studied. We prove that quantum semigroup (and sometimes quantum group) structures arise naturally on such objects out of more fundamental properties. As particular cases we study quantum semigroups of maps preserving a fixed state and quantum commutants of given quantum families of maps.

1. Introduction

Let X be a set. Then the set $\mathrm{Map}(X)$ of all maps $X \to X$ is a semigroup. Of course, the set of all maps fixing a given point of X is a subsemigroup of $\mathrm{Map}(X)$. So is the set of all maps leaving invariant a given measure on X or commuting with a fixed family of maps $X \to X$. These statements border triviality. The situation does not change substantially if we introduce a topology on X and require that all maps be continuous.

Our aim in this paper is to investigate the noncommutative analogs of the above mentioned phenomena. More precisely we will recall the definition of a quantum family of maps between quantum spaces [1] and we will show that, just as in the commutative case mentioned above, quantum semigroup structures appear naturally on many such objects.

In [2] Wang investigated quantum automorphism groups of finite quantum spaces. He searched for universal objects in the category of quantum transformation groups of a given finite quantum space. He also mentioned quantum semigroups in [2, Remark (3), page 208]. We show that the quantum semigroup structure on these objects is there as a consequence of a more fundamental structure these objects possess. This also makes them easier to define.

Wang proved that for nonclassical finite quantum spaces the quantum automorphism group does not exist and turned to study the group preserving a fixed state. We will take a more general approach and define the quantum family of all maps preserving a given state. Again the quantum semigroup structure (and in some cases quantum *group* structure — cf. Section 7) appears from a more fundamental property of these objects.

We will give one more example of a similar situation, where a quantum subsemigroup of a given quantum semigroup is defined without reference to its semigroup structure, by constructing the quantum commutant of a given quantum family

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doi:10.1016/j.geomphys.2008.11.007

of maps. Again, the emergence of a quantum semigroup structure will be a consequence of a more fundamental property of the quantum commutant.

Let us now briefly describe the contents of the paper. Section 2 is a short summary of the standard language of noncommutative topology. In particular we shall recall and discuss the definition of a *quantum space*. In Section 3 we shall define the concept of a quantum family of maps from one quantum space to another. This notion was already introduced in [1], where quantum spaces were called "pseudospaces". We shall define what the *quantum space of all maps* from one quantum space to another is and prove its existence in a special (yet interesting) case. Then we shall define the crucial notion of composition of quantum families of maps.

The quantum space of all maps from a given finite quantum space to itself carries a natural structure of a compact quantum semigroup with unit. This is the content of Section 4. We shall study the properties of this quantum semigroup and its action on the finite quantum space, like ergodicity.

The next two sections are devoted to natural constructions of quantum subsemigroups of the quantum semigroup defined in Section 4. First, in Section 5, we define and study the quantum family of all maps preserving a fixed state. This family is naturally endowed with a compact quantum semigroup structure. The existence of this structure follows from simple considerations concerning composition of quantum families (as defined in Section 3). Then, in Section 6, we construct the quantum commutant of a given quantum family of maps. This construction bears many similarities to the one presented in Section 5. It is based on the notion of *commuting quantum families of maps* which is briefly investigated at the beginning of the section. The quantum commutant has a natural structure of a quantum semigroup.

The constructions presented in Sections 5 and 6 clarify the mechanism of obtaining quantum semigroup structure. This has never been investigated before. In the last section we address the question when quantum *group* structures appear and when they should be expected and show how S. Wang's quantum automorphism groups of finite spaces [2] fit into the framework developed in Section 5.

2. Quantum spaces

Let $\mathfrak{C}$ be the category of C^*-algebras described and studied in [1,3]. The objects of $\mathfrak{C}$ are all C^*-algebras and for any two objects A and B of $\mathfrak{C}$ the set $\mathrm{Mor}(A, B)$ consists of all nondegenerate $*$-homomorphisms from A to $M(B)$. The category $\mathfrak{Q}$ of quantum spaces is by definition the dual category of $\mathfrak{C}$. By definition the class of objects of $\mathfrak{Q}$ is the same as the class of objects of $\mathfrak{C}$. Nevertheless for any C^*-algebra A we shall write $\mathcal{QS}(A)$ for A regarded as an object of $\mathfrak{Q}$.

From the point of noncommutative geometry (topology) it is natural to work with objects of $\mathfrak{Q}$. On the other hand all the tools at our disposal are from the world of C^*-algebras. We shall try to introduce a compromise between the two conventions by declaring that the phrase "let $\mathcal{QS}(A)$ be a quantum space" be taken to mean "let A be a C^*-algebra". Moreover we shall use interchangeably the notation $\Phi \in \mathrm{Mor}(A, B)$ and $\Phi : \mathcal{QS}(B) \to \mathcal{QS}(A)$.

The category of locally compact topological spaces with continuous maps is a full subcategory of $\mathfrak{Q}$. A quantum space $\mathcal{QS}(A)$ is a locally compact space if and only if A is a commutative C^*-algebra. In this case $A = C_\infty(\mathcal{QS}(A))$. In widely accepted terminology quantum spaces corresponding to commutative C^*-algebras are called *classical* spaces.

Many notions from topology are often generalized to the noncommutative setting. As an example let us mention the fact that a quantum space $\mathcal{QS}(A)$ is called *compact* if A is unital. If A is finite dimensional then $\mathcal{QS}(A)$ is said to be a *finite quantum space*. A more controversial idea to call a quantum space $\mathcal{QS}(A)$ a *finite dimensional* if A is finitely generated was proposed in [1].

An interesting step towards a better understanding of the category $\mathfrak{Q}$ was taken in [4,3] (see also [5]). The results of these papers show that any (separable) C^*-algebra is of the form $C_\infty(\mathbb{X})$, where $\mathbb{X}$ is a certain W^*-category and $C_\infty(\,\cdot\,)$ has a whole new meaning (which reduces to the old one for commutative C^*-algebras). This means that, despite technical complications, it is possible to realize quantum spaces as concrete mathematical objects.

3. Quantum families of maps

In this section we shall introduce the objects of our study. These will be quantum spaces of maps or quantum families of maps. The latter concept is a generalization of a classical notion of a continuous family of maps between locally compact spaces labeled by some other locally compact space.

This is based on the fact that for topological spaces X, Y and Z such that Z is Hausdorff and X is locally compact (Hausdorff) we have

$$C(Z \times X, Y) \approx C\left(Z, C(X, Y)\right),$$

where "$\approx$" means homeomorphism and all spaces are taken with compact-open topology [6]. Thus a continuous family of maps $X \to Y$ labeled by Z can be represented by a continuous map $Z \times X \to Y$.

Definition 1. Let $\mathcal{QS}(A)$, $\mathcal{QS}(B)$ and $\mathcal{QS}(C)$ be quantum spaces.

(1) A *quantum family of maps* $\mathcal{QS}(C) \to \mathcal{QS}(B)$ *labeled by* $\mathcal{QS}(A)$ is an element $\Psi \in \mathrm{Mor}(B, C \otimes A)$.
(2) We say that $\Phi \in \mathrm{Mor}(B, C \otimes A)$ is the *quantum family of all maps* $\mathcal{QS}(C) \to \mathcal{QS}(B)$ if for any quantum space $\mathcal{QS}(D)$ and any quantum family Ψ of maps $\mathcal{QS}(C) \to \mathcal{QS}(B)$ labeled by $\mathcal{QS}(D)$ there exists a unique $\Lambda \in \mathrm{Mor}(A, D)$ such