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Expansion of the Lie algebra and its applications ☆

Fukui Guo, Yufeng Zhang *

Information School, Shandong University of Science and Technology, Qingdao Huangdao 266510, PR China

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Abstract

We take the Lie algebra A1 as an example to illustrate a detail approach for expanding a finite dimensional Lie algebra into a higher-dimensional one. By making use of the late and its resulting loop algebra, a few linear isospectral problems with multi-component potential functions are established. It follows from them that some new integrable hierarchies of soliton equations are worked out. In addition, various Lie algebras may be constructed for which the integrable couplings of soliton equations are obtained by employing the expanding technique of the the Lie algebras.

1. Expansions of the Lie algebras

The Lie algebra A_1 and its resulting loop algebra $\tilde{A}_1$ are extensively used to produce integrable hierarchies in soliton theory, because they possess the simple and nontrivial commuting operations. For example, taking the basis of the Lie algebra A_1 as follows:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

the commuting operation is given by

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2}$$

Denoting $a = a_1h + a_2e + a_3f$, $b = b_1h + b_2e + b_3f$, it is easy to calculate that

$$[a, b] = (a_2b_3 - a_3b_2)h + 2(a_1b_2 - a_2b_1)e + 2(a_3b_1 - a_1b_3)f. \tag{3}$$

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* Corresponding author. Tel.: +86 411 81151606.
E-mail address: mathzhang@126.com (Y. Zhang).

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Definition 1. For the two vectors in R^3

$$\bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

a commuting operation is defined by

$$[\bar{a}, \bar{b}] = \begin{pmatrix} a_2b_3 - a_3b_2 \\ 2(a_1b_2 - a_2b_1) \\ 2(a_3b_1 - a_1b_3) \end{pmatrix}. \tag{4}$$

In terms of (3) and (4), it is no doubt that R^3 becomes a Lie algebra, and the Lie algebra A_1 is isomorphic to R^3.

If we take the basis of the Lie algebra A_1 as follows [1]:

$$\bar{h} = 1/2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{\pm} = 1/2\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \tag{5}$$

then

$$[\bar{h}, e_{\pm}] = e_{\mp}, \quad [e_-, e_+] = \bar{h}. \tag{6}$$

As for two elements in A_1 as follows:

$$a = a_1\bar{h} + a_2e + a_3e_-, \quad b = b_1\bar{h} + b_2e_+ + b_3e_-,$$

it is easy to find that

$$[a, b] = (a_3b_2 - a_2b_3)\bar{h} + (a_1b_3 - a_3b_1)e_+ + (a_1b_2 - a_2b_1)e_-. \tag{7}$$

Definition 2. For the two vectors in R^3 in the following:

$$\bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

a commuting operation is given by

$$[\bar{a}, \bar{b}] = \begin{pmatrix} a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix}. \tag{8}$$

We can verify that R^3 also becomes a Lie algebra and it is isomorphic to the Lie algebra A_1. Via denoting $[\bar{a}, \bar{b}]_1$ and $[\bar{a}, \bar{b}]_2$ as the operations (4) and (8), respectively, there is an open problem that how we expand the Lie algebra A_1 into the higher-dimensional Lie algebras with the same simple operational rules as above. We find that such the Lie algebras are various. In the following, the six kinds of operations in a Lie algebra are exhibited by definition.

Definition 3. Let two vectors in R^6 be as follows:

$a = (a_1,a_2,a_3,a_4,a_5,a_6)^{\mathrm{T}}$, $b = (b_1,b_2,b_3,b_4,b_5,b_6)^{\mathrm{T}}$, the following six operations may make R^6 become a Lie algebra

$$[a, b]_1 = \begin{pmatrix} a_2b_3 - a_3b_2 \\ 2(a_1b_2 - a_2b_1) \\ 2(a_3b_1 - a_1b_3) \\ a_5b_6 - a_6b_5 \\ 2(a_4b_5 - a_5b_4) \\ 2(a_6b_4 - a_4b_6) \end{pmatrix}, \tag{9}$$