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A novel model for allometric scaling laws for different organs

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Abstract

The power function of metabolic rate scaling for an organ is established as $B_{\text{organ}} \sim T_{\text{organ}}^{(D+N/6)/(D+1)}$, where B_{organ} is the metabolic rate of the organ, T_{organ} is its mass, D is its fractal dimension of the total cell boundary, and N is its cell's degree of freedom of motion. This prediction agrees quite well with the experimental data for the brain, liver, heart, and kidneys, and it explains very well the reason why the maximal metabolic rate induced by exercise scales with $M^{0.86}$ rather then $M^{0.75}$.

1. Introduction

Allometric scaling laws in biology have attracted considerable attention after the insightful works by West et al. [1,2], Darveau et al. [3], and others [4–10]. Allometric analysis is also a powerful mathematical tool in engineering [11–18]. Wang et al. [19] reconstructed Kleiber's law [20] at the organ–tissue level, and they found that four metabolically active organs, brain, liver, kidneys and heart, have high specific resting metabolic rates when compared with the remaining less-active tissues, such as skeletal muscle, adipose tissue, bone and skin. Brain, liver, kidneys and heart together account for ~60% of resting energy expenditure in humans, even though the four organs represent <6% of body mass. At present and despite decades of concentrated effort, we do not have a rational theory which could explain different scaling laws for different organs. Though there exist dueling theories [21–30] to explain allometric scaling laws in biology, it is intriguing that, as far as the present authors know, no general model in a cell level exists to mechanistically explain scaling laws. A fresh approach, therefore, is still much-needed, and we conclude that cell fractal is the key, and we name the new branch as *cytofractalogy*, which is different from that suggested by West et al. [1,2].

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2. Cytofractalogy: fractal approach to allometrical scaling laws in biology

Now we consider a multicellular organ, and assume that there are n basal cells with characteristic (or typical) radius r. The metabolic rate B of the organ scales linearly with respect to its total surface:

$$B \sim nr^2 = A, \tag{1}$$

where A is the total surface of all cells in the organ. The total surface of cell boundaries is of fractal construction; we, therefore, have

$$A \sim r^D, \tag{2}$$

where D is the fractal dimension of the total cell surface. Due to the smallness of the cells, we can assume that cells are space-filling (thus, for example, the nutrition can reach each cell in the organism). Consequently, the fractal dimension, D, tends to 3, i.e., $D = 3$ for most three-dimensional organs [8].

Combination of (1) and (2) leads to the following scaling relationship:

$$n \sim r. \tag{3}$$

The mass M of the organ scales linearly with respect to its total volume of cells:

$$M \sim nr^3. \tag{4}$$

Note that (4) is not valid for bone tissue. From the scaling relations (1), (3), and (4), we obtain the well-known Kleiber's 3/4 allometric scaling law [8], which reads

$$B \sim M^{3/4}. \tag{5}$$

In 1977, Blum [31] suggested that the 3/4-law can be understood by a four-dimensional approach. In D-dimension space, the "area" A of the hypersurface enclosing an D-dimensional hypervolume scales like $A \sim V^{(D-1)/D}$. When $D = 4$, we have $A \sim V^{3/4}$, a four-dimensional construction:

$$B \sim M^{(D-1)/D}. \tag{6}$$

In view of E1 Naschie's E-infinity theory [32–38], we modify the scaling law (6) in the form

$$B \sim M^{(D_{\mathrm{H}}-1)/D_{\mathrm{H}}} = M^{0.764}, \tag{7}$$

where D_{H} is the expectation value of the Hausdorff dimension of $\varepsilon^{(\infty)}$ [32–38]:

$$D_{\mathrm{H}} = \left\langle \underset{\mathrm{H}}{\mathrm{Dim}}\, \varepsilon^{(\infty)} \right\rangle = \sum_{n=0}^{\infty} n\phi^n = \left(\frac{1}{\phi}\right)^3 = 4.23606\cdots, \tag{8}$$

where $\phi = (\sqrt{5}-1)/2$ is the golden mean, which is the building stone of E1 Naschie's $\varepsilon^{(\infty)}$ network [32–38].

Note that $D = 4$ is the dimension of classical spacetime upon which Einstein's theory is based. This is only an approximation of the true geometry of the universe in the large. Similarly $D = 4$ is used in four-dimensional life [2], and we argue that E1 Naschie's theory might lead to an accurate prediction of metabolic rate. The main application of the E-infinity theory shows miraculous scientific exactness, especially in determining theoretically coupling constants and the mass spectrum of the standard model of elementary particles [32–38].

Now we consider a leaf of a plant or a hepatic cellular plate (see Fig. 1). For an approximate two-dimensional (2-D) organ, the fractal dimension, D, in (2) tends to 2, i.e., $D = 2$ [8]; as a result, we have $nr^2 = A \sim r^2$, leading to the result:

$$n \sim r^0. \tag{9}$$

We, therefore, obtain a 2/3-law for 2-D organs [8]:

$$B \sim M^{2/3}. \tag{10}$$

It is obvious that the Rubner 2/3-law [39] is valid for 2-D lives, and Kleiber's law for 3-D lives as predicted by He and Chen [8].

If the exponent in (10) is linked to the gold mean, then Rubner 2/3-law can be modified as

$$B \sim M^{\phi} = M^{0.618} \tag{11}$$

for two-dimensional organs.

It is interesting to note the scaling relationship $n \sim r$ (see (3)), which is valid for 3-D organs. Remember that r is a space dimension, so the number of cells in an organ endows another *life dimension* [9]. If a cell is isolated from a heart of