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The basic bundle gerbe on unitary groups

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ABSTRACT

We consider the construction of the basic bundle gerbe on $SU(n)$ introduced by Meinrenken and show that it extends to a range of groups with unitary actions on a Hilbert space including $U(n)$ and $U_p(H)$, the Banach Lie group of unitaries differing from the identity by an element of a Schatten ideal. In all these cases we give an explicit connection and curving on the basic bundle gerbe and calculate the real Dixmier–Douady class. Extensive use is made of the holomorphic functional calculus for operators on a Hilbert space.

1. Introduction

Gerbes were introduced by Giraud [16] to study non-abelian cohomology. Brylinski popularised them in his book [4], in particular the case of interest here, which is gerbes with band the sheaf of smooth $U(1)$ valued functions. To every such gerbe is associated a characteristic class in $H^3(M,\mathbb{Z})$, the Dixmier–Douady class of the gerbe. Equivalence classes of gerbes on M are, through this characteristic class, in bijective correspondence with $H^3(M,\mathbb{Z})$. Therefore, it is natural to look for gerbes on manifolds with non-zero degree three integer cohomology. One important example of such a manifold is a compact, simple and simply connected Lie group.

Recall that if G is a compact, simple and simply connected Lie group then $H^3(G,\mathbb{Z}) = \mathbb{Z}$ and there is a canonical closed three-form ν on G — the *basic three-form* (see for instance [23]). ν is a de Rham representative for the image in real cohomology of the generator of $H^3(G,\mathbb{Z})$. The three-form ν is given by

$$\nu = \frac{1}{12}\langle \theta_L, [\theta_L, \theta_L]\rangle$$

where θ_L is the left Maurer–Cartan 1-form on G and $\langle\ ,\ \rangle$ is the *basic* inner product on $\mathfrak{g}$ [23]. In the case where $G = SU(n)$ the basic three-form is

$$-\frac{1}{24\pi^2}\operatorname{tr}(g^{-1}dg)^3. \tag{1.1}$$

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Although the unitary group $U(n)$ is not simply connected, we still have the isomorphism $H^3(U(n), \mathbb{Z}) = \mathbb{Z}$. The three-form (1.1) on $SU(n)$ is clearly the restriction of a closed 3-form defined on $U(n)$. This three-form is the image in real cohomology of the generator of $H^3(U(n), \mathbb{Z})$ — we will refer to it as the basic three-form on $U(n)$.

The basic three-form ν was exploited to good effect in Witten's paper [28] on WZW models. Witten considered a non-linear sigma-model in which the fields of the theory were smooth maps $g: \Sigma \to G$ from a compact Riemann surface Σ to a compact, simple and simply connected Lie group G. In constructing a conformally invariant action for this sigma-model Witten was lead to consider the Wess-Zumino term

$$S_{WZ}(g) = \int_B \tilde{g}^* \nu.$$

Here B is a 3-manifold with boundary Σ and $\tilde{g}: B \to G$ is an extension of g to B. The question arises as to whether the Wess-Zumino term is well defined. It turns out that under these topological assumptions on G, one can always find such an extension of g, and, due to the integrality property of the basic three-form ν, $\exp(2\pi i S_{WZ}(g))$ is well defined. It is natural to wonder if it is possible to remove the topological assumptions on G and make sense of this action when G is a non-simply connected group. The theory of gerbes provides a valuable way of thinking about this problem (see for example [25]). One can interpret the action $\exp(2\pi i S_{WZ}(g))$ as the holonomy over Σ of a canonically defined gerbe on G — the *basic gerbe*. This basic gerbe on G can be constructed even when the group G is not simply connected. Since the holonomy of a gerbe on a manifold can be defined irrespective of whether the manifold is simply connected or not, we see that by *defining* the action to be the holonomy of the basic gerbe over Σ, we can remove this topological assumption on the group G.

There have been a number of constructions of gerbes and bundle gerbes on a Lie group G in the literature since Brylinksi's book [4] appeared and we review them briefly to put the results of this paper into perspective. Indeed the first such construction appeared in [4] (and later in [5]); it involved the path-fibration of G and thus was inherently infinite-dimensional. It was pointed out in [4] that it would be interesting to have a finite dimensional construction.

The notion of bundle gerbe was introduced by the first author in [19]. The relationship of bundle gerbes with gerbes is analogous to that between line bundles thought of as fibrations and line bundles thought of as locally free sheaves of modules. Bundles gerbes correspond to fibrations of groupoids whereas gerbes involve sheaves of groupoids. In [19] the tautological bundle gerbe was introduced. This was a bundle gerbe associated to any integral, closed three-form on a 2-connected manifold M. This implicitly includes the case of compact, simple, simply-connected Lie groups which was discussed more explicitly in [11]. Again these constructions are infinite-dimensional and related to the path fibration. There is a simple way of defining this bundle gerbe using the so-called lifting bundle gerbe described in [19]. The path-fibration over G is a principal bundle with structure group ΩG, the group of based loops in G, and there is a well known central extension $\widehat{\Omega G}$ of ΩG by the circle (see [23]) which one can use to form a bundle gerbe. This bundle gerbe measures the obstruction to lifting the path-fibration to a bundle with structure group $\widehat{\Omega G}$.

The next construction, due to Brylinski [6] (see also [7]), involves the Weyl map

$$G/T \times T \to G$$
$$(gT, t) \mapsto gtg^{-1}.$$

A gerbe was defined on $G/T \times T$ using a 'cup product' construction involving line bundles on G/T and functions on T. It was shown, using some delicate sheaf arguments, that this gerbe pushes forward via the Weyl map to a gerbe on G. Brylinski notes that this construction gives an equivariant gerbe for the conjugation action of G on G. This construction of Brylinski seems to be the most general construction to date, however the geometry of this gerbe has not been explored in full detail.

Following this construction of Brylinski's was a construction of Gawedzki and Reis [14] for the case when $G = SU(n)$. The case of quotients of $SU(n)$ by finite subgroups of the centre was also treated. The methods used in this construction involved ideas from representation theory. Gawedzki and Reis also defined a connection and curving on their bundle gerbe. This work was followed shortly afterwards by a paper of Meinrenken [18]. This gave a definitive treatment of the case where G was an arbitrary compact, simple and simply connected Lie group. This construction was also representation theoretic in nature and involved the structure of the sets of regular and singular elements of G. Meinrenken's paper also gave an extensive discussion of equivariant bundle gerbes; the basic bundle gerbe constructed in the paper was shown to be equivariant and equipped with an equivariant connection and curving. A simpler construction of a local bundle gerbe in the sense of Chatterjee–Hitchin was also given for the case of $G = SU(n)$ — we shall comment more on this below.

Equivariant gerbes were also studied in the paper [3] of Behrend, Xu and Zhang; the authors constructed a bundle gerbe using the path-fibration and equipped it with an equivariant connection and curving. This construction was followed shortly by a paper of Gawedski and Reis [15] giving a generalisation of Meinrenken's construction to the case of non-simply connected groups.

Finally we come to the case of interest in this paper which is the construction of a local bundle gerbe over $SU(n)$. As mentioned above this example first appeared in the paper [18] of Meinrenken and was later discussed also by Mickelsson [21]. In this construction, a local bundle gerbe was defined over the open cover of $SU(n)$ by open sets U_z consisting of unitary matrices for which z is not an eigenvalue.

We will show how to remove the dependence on the local cover in Meinrenken's and Mickelsson's construction and to generalise it to any group G, which is one of the following: the unitary group $U(n)$ (more generally the group $U(H)$ of unitary operators on some finite dimensional Hilbert space H), the (diagonal) torus $T = \mathbb{T}^n \subset U(n)$, or one of the Banach