Journal of Geometry and Physics 58 (2008) 1063–1079

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Review

Prym varieties and applications

A. Lesfari *

Department of Mathematics, Faculty of Sciences, University of Chouaïb Doukkali, B.P. 20, El-Jadida, Morocco

ARTICLE INFO

Article history:
Received 7 March 2007
Received in revised form 5 April 2008
Accepted 7 April 2008
Available online 15 April 2008

JGP SC:
Algebraic geometry
Classical integrable systems

MSC:
14Q05
14H40
70H06

Keywords:
Curves
Prym varieties
Integrable systems

ABSTRACT

The classical definition of Prym varieties deals with the unramified covers of curves. The aim of this article is to give explicit algebraic descriptions of the Prym varieties associated with ramified double covers of algebraic curves. We make a careful study of the connection with the concept of algebraic completely integrable systems and we apply the methods to some problems such as the Hénon–Heiles system, the Kowalewski rigid body motion and Kirchhoff's equations of motion of a solid in an ideal fluid.

Contents

1. Introduction

During the last decades, algebraic geometry has become a tool for solving differential equations and spectral questions of mechanics and mathematical physics. This paper consists of two separate but related topics: the first part purely algebraic-geometric, the second one on Prym varieties in algebraic integrability.

Prym variety $\text{Prym}(\mathcal{C}/\mathcal{C}_0)$ is a subabelian variety of the jacobian variety $\text{Jac}(\mathcal{C}) = \text{Pic}^0(\mathcal{C}) = H^1(\mathcal{O}_C)/H^1(\mathcal{C},\mathbb{Z})$ constructed from a double cover $\mathcal{C}$ of a curve $\mathcal{C}_0$: if σ is the involution on C interchanging sheets, then σ extends by

* Corresponding address: B.P. 271, El-Jadida, Morocco.
E-mail addresses: Lesfariahmed@yahoo.fr, lesfari@ucd.ac.ma.

0393-0440/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2008.04.005

linearity to a map $\sigma : \text{Jac}(\mathcal{C}) \longrightarrow \text{Jac}(\mathcal{C})$. Up to some points of order two, $\text{Jac}(\mathcal{C})$ splits into an even part and an odd part: the even part is $\text{Jac}(\mathcal{C}_0)$ and the odd part is a $\text{Prym}(\mathcal{C}/\mathcal{C}_0)$. The classical definition of Prym varieties deals with the unramified double covering of curves and was introduced by W. Schottky and H. W. E. Jung in relation with the Schottky problem [6] of characterizing jacobian varieties among all principally polarized abelian varieties (an abelian variety is a complex torus that can be embedded into projective space). The theory of Prym varieties was dormant for a long time, until revived by D. Mumford around 1970. It now plays a substantial role in some contemporary theories, for example integrable systems [1,2,5,8,10,14,15,17], the Kadomtsev–Petviashvili equation (KP equation), in the deformation theory of two-dimensional Schrödinger operators [19], in relation to Calabi–Yau three-folds and string theory, in the study of the generalized theta divisors on the moduli spaces of stable vector bundles over an algebraic curve [7,11],…

Integrable hamiltonian systems are nonlinear ordinary differential equations described by a hamiltonian function and possessing sufficiently many independent constants of motion in involution. By the Arnold–Liouville theorem [4,16], the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following differential geometric fact; a compact and connected n-dimensional manifold on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus and each vector field will define a linear flow there. A dynamical system is algebraic completely integrable (in the sense of Adler–van Moerbeke [1]) if it can be linearized on a complex algebraic torus $\mathbb{C}^n/lattice$ (=abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines.

One of the remarkable developments in recent mathematics is the interplay between algebraic completely integrable systems and Prym varieties. The period of these Prym varieties provide the exact periods of the motion in terms of explicit abelian integrals. The aim of the first part of the present paper is to give explicit algebraic descriptions of the Prym varieties associated to ramified double covers of algebraic curves. The basic algebraic tools are known and can be found in the book by Arbarello, Cornalba, Griffiths, Harris [3] and in Mumford's paper [18]. In the second part of the paper, we make a careful study of the connection with the concept of algebraic completely integrable systems and we apply the methods to some problems such as the Hénon–Heiles system, the Kowalewski rigid body motion and Kirchhoff's equations of motion of a solid in an ideal fluid. The motivation was the excellent Haine's paper [10] on the integration of the Euler–Arnold equations associated to a class of geodesic flow on $SO(4)$. The Kowalewski's top and the Clebsch's case of Kirchhoff's equations describing the motion of a solid body in a perfect fluid were integrated in terms of genus two hyperelliptic functions by Kowalewski [13] and Kötter [12] as a result of complicated and mysterious computations. The concept of algebraic complete integrability (Adler–van Moerbeke) throws a completely new light on these two systems. Namely, in both cases (see [14] for Kowalewski's top and [10] for geodesic flow on $SO(4)$), the affine varieties in $\mathbb{C}^6$ obtained by intersecting the four polynomial invariants of the flow are affine parts of Prym varieties of genus 3 curves which are double cover of elliptic curves. Such Prym varieties are not principally polarized and so they are not isomorphic but only isogenous to Jacobi varieties of genus two hyperelliptic curves. This was a total surprise as it was generically believed that only jacobians would appear as invariants manifolds of such systems. The method that is used for reveiling the Pryms is due to Haine [10]. By now many algebraic completely integrable systems are known to linearize on Prym varieties.

2. Prym varieties

Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}_0$ be a double covering with $2n$ branch points where $\mathcal{C}$ and $\mathcal{C}_0$ are nonsingular complete curves. Let $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be the involution exchanging sheets of $\mathcal{C}$ over $\mathcal{C}_0 = \mathcal{C}/\sigma$. By Hurwitz's formula, the genus of $\mathcal{C}$ is $g = 2g_0 + n - 1$, where g_0 is the genus of $\mathcal{C}_0$. Let $\mathcal{O}_{\mathcal{C}}$ be the sheaf of holomorphic functions on $\mathcal{C}$. Let $S^g(\mathcal{C})$ be the g-th symmetric power of $\mathcal{C}$ (the totality of unordered sets of points of $\mathcal{C}$). On $S^g(\mathcal{C})$, two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly equivalent (in short, $\mathcal{D}_1 \equiv \mathcal{D}_2$) if their difference $\mathcal{D}_1 - \mathcal{D}_2$ is the divisor of a meromorphic function or equivalently if and only if $\int_{\mathcal{D}_1}^{\mathcal{D}_2} \omega = \int_\gamma \omega, \forall \omega \in \Omega^1_{\mathcal{C}}$, where $\Omega^1_{\mathcal{C}}$ is the sheaf of holomorphic 1-forms on $\mathcal{C}$ and γ is a closed path on $\mathcal{C}$. We define the jacobian (Jacobi variety) of $\mathcal{C}$, to be $\text{Jac}(\mathcal{C}) = S^g(\mathcal{C})/\equiv$. To be precise, the jacobian $\text{Jac}(\mathcal{C}) = \text{Pic}^0(\mathcal{C})$ of $\mathcal{C}$ is the connected component of its Picard group parametrizing degree 0 invertible sheaves. It is a compact commutative algebraic group, i.e., a complex torus. Indeed, from the fundamental exponential sequence, we get an isomorphism

$$\text{Jac}(\mathcal{C}) = \text{Pic}^0(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})/H_1(\mathcal{C}, \mathbb{Z}) \simeq H^0(\mathcal{C}, \Omega^1_{\mathcal{C}})^*/H_1(\mathcal{C}, \mathbb{Z}) \simeq \mathbb{C}^g/\mathbb{Z}^{2g},$$

via the Serre's duality. Consider the following mapping

$$S^g(\mathcal{C}) \longrightarrow \mathbb{C}^g/L_\Omega, \qquad \sum_{k=1}^{g} \mu_k \longmapsto \sum_{k=1}^{g} \int^{\mu_k(t)} (\omega_1, \ldots, \omega_g) = t\left(k_1, \ldots, k_g\right),$$

where $(\omega_1, \ldots, \omega_g)$ is a basis of $\Omega^1_{\mathcal{C}}$, L_Ω is the lattice associated to the period matrix Ω and $\mu_1, \ldots, \mu_g$ some appropriate variables defined on a non empty Zariski open set. Let $(a_1, \ldots, a_{g_0}, \ldots, a_g, b_1, \ldots, b_{g_0}, \ldots, b_g)$, be a canonical homology basis of $H_1(\mathcal{C}, \mathbb{Z})$ such that $\sigma(a_1) = a_{g_0+n}, \ldots, \sigma(a_{g_0}) = a_g, \sigma(a_{g_0+1}) = -a_{g_0+1}, \ldots, \sigma(a_{g_0+n-1}) = -a_{g_0+n-1}, \sigma(b_1) = b_{g_0+n}, \ldots, \sigma(b_{g_0}) = b_g, \sigma(b_{g_0+1}) = -b_{g_0+1}, \ldots, \sigma(b_{g_0+n-1}) = -b_{g_0+n-1}$, for the involution σ. Notice