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Extended studies of separability functions and probabilities and the relevance of Dyson indices

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ABSTRACT

We report substantial progress in the study of *separability functions* and their application to the computation of *separability probabilities* for the real, complex and quaternionic qubit–qubit and qubit–qutrit systems. We expand our recent work [P.B. Slater, J. Phys. A 39 (2006) 913], in which the Dyson indices of random matrix theory played an essential role, to include the use of not only the volume element of the Hilbert–Schmidt (HS) metric, but also that of the Bures (minimal monotone) metric as measures over these finite-dimensional quantum systems. Further, we now employ the Euler-angle parameterization of density matrices (ρ), in addition to the Bloore parameterization. The Euler-angle separability function for the minimally degenerate complex two-qubit states is well-fitted by the sixth-power of the participation ratio, $R(\rho) = \frac{1}{\mathrm{Tr}\rho^2}$. Additionally, replacing $R(\rho)$ by a simple linear transformation of the Verstraete–Audenaert–De Moor function [F. Verstraete, K. Audenaert, B.D. Moor, Phys. Rev. A 64 (2001) 012316], we find close adherence to Dyson-index behaviour for the real and complex (nondegenerate) two-qubit scenarios. Several of the analyses reported help to fortify our conjectures that the HS and Bures separability probabilities of the complex two-qubit states are $\frac{8}{33} \approx 0.242424$ and $\frac{1680(\sqrt{2}-1)}{\pi^8} \approx 0.733389$, respectively. Employing certain *regularized beta functions* in the role of Euler-angle separability functions, we closely reproduce – consistently with the Dyson-index *ansatz* – several HS two-qubit separability probability conjectures.

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1. Introduction

For several years now, elaborating upon an idea proposed in [3], we have been pursuing the problem of deriving (hypothetically exact) formulas for the proportion of states of qubit–qubit [4] and qubit–qutrit [5] systems that are *separable* (classically-correlated) in nature [1,6–12]. Of course, any such proportions will critically depend on the measure that is placed upon the quantum systems. In particular, we have – in analogy to (classical) Bayesian analyses, in which the *volume element* of the *Fisher information* metric for a parameterized family of probability distributions is utilized as a measure ("Jeffreys' prior") [13] – principally employed the volume elements of the well-studied (Euclidean, flat) Hilbert–Schmidt (HS) and Bures (*minimal* monotone or symmetric-logarithmic-derivative [SLD]) metrics (as well as a number of other [non-minimal] *monotone* metrics [10]).

Życzkowski and Sommers [14,15] have, using methods of random matrix theory [16] (in particular, the Laguerre ensemble), obtained formulas, general for all n, for the HS and Bures *total* volumes (and hyperareas) of $n \times n$ (real and complex) quantum systems. Up to normalization factors, the HS total volume formulas were also found by Andai [17], in a rather different analytical framework, using a number of (spherical and beta) integral identities and positivity (Sylvester) conditions. (He also obtained formulas – general for any monotone metric [including the Bures] – for the volume of *one*-qubit [$n = 2$] states [17, Section 4].)

Additionally, Andai did specifically study the HS *quaternionic* case. He derived the HS total volume for $n \times n$ quaternionic systems [17, p. 13646],

$$V^{\mathrm{HS}}_{\mathrm{quat}} = \frac{(2n-2)!\pi^{n^2-n}}{(2n^2-n-1)!} \prod_{i=1}^{n-2} (2i)!, \tag{1}$$

giving us for the two-qubit ($n = 4$) case that will be our specific initial interest here, the 27-dimensional volume,

$$\frac{\pi^{12}}{7776000} \cdot \frac{1}{40518448303132800} = \frac{\pi^{12}}{315071454005160652800000} \approx 2.93352 \times 10^{-18}. \tag{2}$$

(In the analytical setting employed by Życzkowski and Sommers [14], this volume would appear as 2^{12} times as large [17, p. 13647].)

If one then possessed a companion volume formula for the *separable* subset, one could immediately compute the HS two-qubit quaternionic separability *probability* ($P^{\mathrm{HS}}_{\mathrm{quat}}$) by taking the ratio of the two volumes. In fact, following a convenient paradigm we have developed, and will employ several times below, in varying contexts, we will compute $P^{\mathrm{HS}}_{\mathrm{quat}}$ as the product ($R_1 R_2$) of two *ratios*, R_1 and R_2. The first (24-dimensional) factor on the left-hand side of (2) will serve as the denominator of R_1 and the second (3-dimensional) factor, as the denominator of R_2. The determinations of the *numerators* of such pairs of complementary ratios will constitute, in essence, our (initial) principal computational challenges.

1.1. Bloore parameterization of density matrices

One analytical approach to the separable volume/probability question that has recently proved to be productive [2] – particularly, in the case of the Hilbert–Schmidt (HS) metric (cf. [18]) – makes fundamental use of a (quite elementary) form of density matrix parameterization first proposed by Bloore [19]. This methodology can be seen to be strongly related to the very common and long-standing use of *correlation matrices* in statistics and its many fields of application [20–22]. (Correlation matrices can be obtained by standardizing *covariance* matrices. Density matrices have been viewed as covariance matrices of multivariate normal [Gaussian] distributions [23]. Covariance matrices for certain observables have been used to study the separability of finite-dimensional quantum systems [24]. The possible states of polarization of a two-photon system are describable by six Stokes parameters and a 3×3 "polarization correlation" matrix [25].)

In the Bloore (off-diagonal scaling) parameterization, one simply represents an off-diagonal ij-entry of a density matrix ρ, as $\rho_{ij} = \sqrt{\rho_{ii}\rho_{jj}} w_{ij}$, where w_{ij} might be real, complex or quaternionic [26–28] in nature. The particular attraction of the Bloore scheme, in terms of the separability problem in which we are interested, is that one can (in the two-qubit case) implement the well-known Peres–Horodecki separability (positive-partial-transpose) test [29,30] using only the ratio,

$$\mu = \sqrt{\nu} = \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}, \tag{3}$$

rather than the four (three independent) diagonal entries of ρ individually [1, Eq. (7)], [2, Eq. (5)].

Utilizing the Bloore parameterization, we have, accordingly, been able to reduce the problem of computing the desired HS volumes of two-qubit separable states to the computations of *one*-dimensional integrals (33) over $\mu \in [0, \infty]$. The associated integrands are the *products* of *two* functions, one a readily determined jacobian function $\mathcal{J}(\mu)$ (corresponding, first, to the transformation to the Bloore variables w_{ij} and, then, to μ) and the other, the more problematical (what we have termed) *separability function* $\mathcal{S}^{\mathrm{HS}}(\mu)$ [1, Eqs. (8), (9)].

In the qubit-*qutrit* case [Section 2.3], *two* ratios,

$$\nu_1 = \frac{\rho_{11}\rho_{55}}{\rho_{22}\rho_{44}}, \qquad \nu_2 = \frac{\rho_{22}\rho_{66}}{\rho_{33}\rho_{55}}, \tag{4}$$