Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 2024–2048
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Definite signature conformal holonomy: A complete classification

Stuart Armstrong*

Erwin Schrödinger Institute, Vienna, Austria

Received 18 September 2006; received in revised form 10 May 2007; accepted 10 May 2007
Available online 16 May 2007

Abstract

This paper aims to classify the holonomy of the conformal Tractor connection, and relate these holonomies to the geometry of the underlying manifold. The conformally Einstein case is dealt with through the construction of metric cones, whose Riemannian holonomy is the same as the Tractor holonomy of the underlying manifold. Direct calculations in the Ricci-flat case and an important decomposition theorem complete the classification for definitive signature.

Keywords: Conformal geometry; Cartan connection; Tractor connection; Holonomy; Einstein manifold; Cone

1. Introduction

Conformal geometry is perhaps the most natural extension of Riemannian geometry, and shares many of the same features with it. However, it was realised early on – as far back as Cartan [16] – that one of the most mathematically rewarding ways of dealing with conformal geometry was not by analogy with Riemannian geometries, but by analogy with the other parabolic geometries, using the general *Cartan connection* as a universal tool.

These parabolic geometries are a class of geometries that include, amongst others, projective, almost Grassmannian, almost quaternionic, and codimension one CR structures. The common point of these is that their 'flat' model space is the Lie group quotient G/P, where P is parabolic. Papers [1,2] by the same author deals with the projective case, while this paper treats the conformal one.

Many figures contributed to understanding parabolic geometries; Thomas [34,35] developed key ideas for Cartan connection calculus, and Shiego Sasaki investigated the conformal case in 1943 [32,33], followed by Tanaka [31] in 1979 and the major paper of Bailey, Eastwood and Gover in 1994 [4].

Since then, there have been a series of papers by Čap and Gover [14,13,20,15], developing a lot of the techniques that will be used in the present paper. Previous papers had focused on seeing the Cartan connection for conformal geometry as a property of a principal bundle $\mathcal{P}$. More recently, the principal bundle is replaced by an associated vector bundle, the *Tractor bundle* $\mathcal{T}$, and the Cartan connection by a connection form for $\mathcal{T}$, the *Tractor connection* $\overrightarrow{\nabla}$. With these tools, calculations are considerably simplified.

The purpose of this paper is to analyse one of the invariants of the Tractor connection, the holonomy group. There is an invariant metric of signature $(n+1, 1)$ on $\mathcal{T}$, so this holonomy group must be a subgroup of $G = SO(n+1, 1)$.

* Tel.: +44 77 59397688.
E-mail address: dragondreaming@gmail.com.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2007.05.001

It is a well known fact that a parallel section of the Tractor bundle corresponds to the local existence of an Einstein metric in the conformal class of a manifold. Beyond this, little was known about reductions of holonomy.

In this paper, we shall classify all the possible local holonomy groups of $\overrightarrow{\nabla}$ acting *reducibly* on $\mathcal{T}$. In doing so, they must conserve a Lorentzian metric of signature $(n+1,1)$. Then a paper by Di Scala and Olmos [17] shows that we have the complete list: there exist no connected proper subgroups of $SO(n+1,1)$ acting irreducibly on $\mathbb{R}^{n+1,1}$.

Proposition 1.1. *There are no local holonomy algebras acting irreducibly on $\mathcal{T}$ apart from the full $\mathfrak{so}(n+1,1)$ algebra.*

A very recent paper by Felipe Leitner, [28], proves the same results as in this paper; but his methods, involving normal Killing forms, are different from those described here.

The classification comes in two main pieces; if a bundle of rank other than 1 or n is preserved, the manifold decomposes analogously to the De Rham decomposition:

Theorem 1.2. *Let $(M,[g])$ be a conformal, n-dimensional manifold, such that $\mathcal{T}_M$ has a holonomy preserved subbundle of rank k, $2 \leq k \leq n$. Then locally there exists a metric $g \in [g]$ such that (M,g) splits locally into the direct product of two Einstein manifolds N_1, N_2 of dimensions $l = k-1$ and $n-l$. The Einstein constants a and b of N_1, N_2 are related by $(n-l-1)a = (1-l)b$. Furthermore, there are canonical inclusions of the Tractor bundles of N_1 and N_2 into $\mathcal{T}_M$ and the Tractor holonomy group of M is locally the direct product of those of N_1 and N_2.*

That last statement requires a bit of explaining, since the Tractor bundles of N_1 and N_2 are of rank $l+2$ and $n-l+2$ respectively. However, since these are both Einstein manifolds, the effective rank of their Tractor bundles are $l+1$ and $n-l+1$, allowing the decomposition.

The converse to Theorem 1.2 is also true. This decomposition is a local result, and may become degenerate along some embedded submanifolds.

The second step is to list all the possible Tractor holonomies for a conformally Einstein manifold. Using a metric cone construction, related to the ambient metric of [18,15,21], the following list is established:

Theorem 1.3 (*Einstein Classification*). *The Tractor holonomy of $(M^n,[g])$, for M^n conformal to an Einstein space of non-zero scalar curvature, is one of the following groups:*

- $SO(n,1), n \geq 4$,
- $SO(n+1), n \geq 4$,
- $SU(m)$ *for* $2m = n+1, n \geq 4$,
- $Sp(m)$ *for* $4m = n+1$,
- G_2 *for* $n = 6$,
- $\mathrm{Spin}(7)$ *for* $n = 7$.

Moreover, all these actually occur as holonomy groups.

The Ricci-flat case must be treated differently; in fact, if (M^n,g) is Ricci-flat and conformally indecomposable, and G is the metric holonomy group of ∇^g, then $(M^n,[g])$ has Tractor holonomy $G \rtimes \mathbb{R}^n$. Thus:

Theorem 1.4. *The possible indecomposable Tractor holonomy groups for the conformal manifold $(M^n,[g])$, conformally Ricci-flat, are:*

- $SO(n) \rtimes \mathbb{R}^n, n \geq 4$,
- $SU(m) \rtimes \mathbb{R}^{2m}, m \geq 2$,
- $Sp(m) \rtimes \mathbb{R}^{4m}, m \geq 1$,
- $G_2 \rtimes \mathbb{R}^7$,
- $\mathrm{Spin}(7) \rtimes \mathbb{R}^8$,

and all of these groups do occur.