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Geometrical and topological aspects of Electrostatics on Riemannian manifolds

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Abstract

We study some geometrical and topological properties of the electric fields created by point charges on Riemannian manifolds from the viewpoint of the theory of dynamical systems. We provide a thorough description of the structure of the basin boundary and its connection with the topology of the manifold, and characterize the spaces in which the orbits of the electric field are geodesics. We also consider symmetries of electric fields on manifolds, particularly on spaces of constant curvature.

Keywords: Green function; Vector field; Basin boundary; Geodesics; Symmetries

1. Introduction

The discovery of the inverse-square law for Newtonian and Coulomb interactions is a milestone in the physics of the seventeenth and eighteenth centuries. The central claim of electrostatic theory [2,24] is that the force per unit charge experienced by a test particle situated at a point $x \in \mathbb{R}^3$ subject to the interaction created by a charge of magnitude $q \in \mathbb{R}$ is given by the electric vector field

$$E = \frac{q}{4\pi} \frac{x - x_0}{|x - x_0|^3}.$$

Here $x_0 \in \mathbb{R}^3$ is the position of the point particle originating the interaction, and we have chosen Heaviside–Lorentz units. The same law also holds for the gravitational interaction created by a point mass of magnitude $-q$ in natural units.

Since then, the study of electric fields generated by N point charges q_i ($i = 1, \dots, N$) in Euclidean space has become a classical problem in mathematical physics and potential theory [11]. When the charges are all negative, this is equivalent to studying the Newtonian gravitational field created by N point masses $|q_i|$, which also coincides with the first-order approximation to the gravitational field in general relativity [37]. In modern treatments, one usually

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defines the potential function $V_p : \mathbb{R}^3 \to \mathbb{R}$ of a point charge, which is a fundamental solution of the Poisson equation

$$-\Delta V_p = \delta_p,$$

and obtains the electric field as $E = -\nabla V_p$. Here and in what follows, δ_p stands for the Dirac distribution centered at p. The electric field created by several charges can be calculated using the superposition principle.

A natural generalization of this problem is the study of the electric fields generated by point charges on Riemannian spaces. There is a vast literature on the fundamental solutions of the Poisson equation on manifolds, e.g., on the existence of positive fundamental solutions [33,9,17,29,30], the study of upper and lower estimates for these functions [45,31,22], and the connection of these fundamental solutions with the heat kernel [51,32,18].

Nevertheless, the geometric and topological properties of the gradient of the fundamental solutions have received comparatively little attention. In this paper we shall focus on the study of this aspect using techniques from the theory of dynamical systems, and we shall show some interesting connections between the orbits of the electric field (historically known as electric lines or lines of force) and the topology of the space. Thus the concept of electric line, as Faraday used to visualize the electric fields in the nineteenth century, is profitably extended to the framework of general Riemannian manifolds.

Let us sketch the organization of this paper. In Section 2 we define the concepts of Li–Tam fundamental solution, basin boundary, and some other objects of which we make extensive use in the following sections. In Section 3 the topological structure of the electric lines and the basin boundary in an n-manifold is studied, whereas in Section 4 we provide stronger results which hold for electric fields on surfaces ($n = 2$). Section 5 concentrates on the relationship between electric lines and geodesics. In Section 6 we study the symmetries of the electric field and their application to spaces of constant curvature, obtaining some exact solutions. Most of the material in Sections 3–6 is new, including a detailed description of the topological structure of the basin boundary, and a complete characterization of spaces in which the electric lines are geodesics.

2. Definitions

Let (M, g) be a Riemannian n-manifold without boundary, which we shall assume to be open, complete, analytic, connected, finitely generated (i.e., all the homotopy groups of M have finite rank), and such that all its ends are collared. For an arbitrary point $p \in M$, let V_p be a fundamental solution of the Poisson equation

$$-\Delta V_p = \delta_p, \tag{1}$$

Δ standing for the Laplace–Beltrami operator. Here δ_p denotes the Dirac distribution centered at p.

Li and Tam [29] have provided a geometric construction of solutions to this equation for any Riemannian manifold (M, g). Their technique consists in considering a monotone sequence of compact domains $p \in M_1 \subset M_2 \subset \cdots$ which exhaust M, and studying the Dirichlet problem

$$-\Delta V_p^{(k)} = \delta_p \quad \text{in } M_k$$
$$V_p^{(k)} = 0 \quad \text{on } \partial M_k$$

in each M_k. Then a solution to Eq. (1) can be obtained as

$$V_p(x) = \lim_{k\to\infty} V_p^{(k)}(x) - c_k$$

for some sequence of non-negative constants (c_k). The construction guarantees that V_p is analytic in $M - p$, and that it is decreasing in the sense that for all $R > 0$

$$\sup_{M-B_p(R)} V_p = \max_{\partial B_p(R)} V_p,$$

where $B_p(R) = \{x \in M : \mathrm{dist}(x, p) < R\}$. These two properties are key to most of our work in the following sections. Furthermore, the map $v : M \times M \to \mathbb{R}$ given by $v(x, y) = V_y(x)$ is symmetric, and analytic in $\{(x, y) \in M \times M : x \neq y\}$.

When $\inf V_p = -\infty$, V_p is called a non-positive Green function, or an Evans function. This condition only depends on the end structure of (M, g), and when it holds (M, g) is called parabolic. When $\inf V_p > -\infty$, one says that