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Third group cohomology and gerbes over Lie groups

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ABSTRACT

The topological classification of gerbes, as principal bundles with the structure group the projective unitary group of a complex Hilbert space, over a topological space H is given by the third cohomology $\mathrm{H}^3(H, \mathbb{Z})$. When H is a topological group the integral cohomology is often related to a locally continuous (or in the case of a Lie group, locally smooth) third group cohomology of H. We shall study in more detail this relation in the case of a group extension $1 \to N \to G \to H \to 1$ when the gerbe is defined by an abelian extension $1 \to A \to \hat{N} \to N \to 1$ of N. In particular, when $\mathrm{H}^1_s(N, A)$ vanishes we shall construct a transgression map $\mathrm{H}^2_s(N, A) \to \mathrm{H}^3_s(H, A^N)$, where $A^{\hat{N}}$ is the subgroup of N-invariants in A and the subscript s denotes the locally smooth cohomology. Examples of this relation appear in gauge theory which are discussed in the paper.

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1. Introduction

A gerbe over a topological space X is a principal bundle with fiber $PU(\mathcal{H})$, the projective unitary group $U(\mathcal{H})/S^1$ of a complex Hilbert space $\mathcal{H}$. Gerbes are classified by the integral cohomology $\mathrm{H}^3(X, \mathbb{Z})$. In real life gerbes most often come in the following way. We have a vector bundle E over a space Y with model fiber $\mathcal{H}$ with a free right group action $Y \times N \to Y$ with $Y/N = X$ such that the action of N can be lifted to a projective action of N on E. Then E/N becomes a projective vector bundle over X with structure group $PU(\mathcal{H})$. Put in the groupoid language, we have an S^1-extension of the transformation groupoid $Y \times N \to Y$; the circle extension can be written in terms of a 2-cocycle $c_2(y; g, g') \in S^1$ with $y \in Y$ and $g, g' \in N$. In general, there might be topological obstructions to define c_2 as a globally continuous cocycle on N. A well known example of this is when $N = LK$, the smooth loop group of a compact simple Lie group K. This group has nontrivial central S^1-extensions parametrized by $\mathrm{H}^2(N, \mathbb{Z})$ but because these are nontrivial circle bundles over LK the 2-cocycle c_2 is only locally smooth, in a neighborhood of the neutral element.

More complicated examples arise in gauge theory. There N is the group of smooth gauge transformations acting on the space Y of gauge connections and in the quantum theory one is forced to deal with extensions of N with the abelian group A of circle valued functions on Y.

As pointed out by Neeb [1] the abelian extensions, even in the topologically nontrivial case $1 \to A \to \hat{N} \to N \to 1$ of an A-bundle over N, can be constructed in terms of locally smooth 2-cocycles. The natural question is then to which extent

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gerbes over a Lie group H are described by locally smooth 3-cocycles on H. We shall concentrate on the following situation. Let G be an (infinite-dimensional) Lie group, N a normal subgroup of G, $H = G/N$ and A a G-module. Given an extension $\widehat{N}$ of N by A (viewed as an abelian group) described by a representative of an element $c_2 \in \mathrm{H}^2_s(N, A)$, i.e., by a locally smooth 2-cocycle with values in A, we want to define a transgression $\tau(c_2) \in \mathrm{H}^3_s(H, A^N)$, where A^N is the set of N invariant elements in A. This we achieve by assuming that the first cohomology group $\mathrm{H}^1_s(N, A)$ vanishes (cf. Theorem 4.6). In addition, under the same restriction, we show that the transgressed element $\tau(c_2)$ vanishes if and only if there is a prolongation of the group extension $1 \to N \to G \to H \to 1$ to an extension $1 \to \widehat{N} \to \widehat{G} \to H \to 1$ (cf. Theorem 4.7).

A gerbe over H can be always trivialized in the pull-back on G if $\mathrm{H}^3(G, \mathbb{Z}) = 0$. In particular, we can take G as the path group

$$PH = \{f : [0, 1] \to H : f(0) = 1\},$$

which is contractible and is a model for $E\Omega H$, the total space of the universal classifying bundle for the based loop group ΩH with $B\Omega H = H$. Here we could work either in the smooth or continuous category of maps. In the smooth setting this case is actually relevant in gauge theory applications which we shall discuss in the last section. For example, when H is a compact Lie group, then the smooth version of PH can be identified with the space of gauge connections of an H-bundle over the unit circle through the map $f \mapsto f^{-1}df$ and ΩH (assuming periodic boundary conditions for $f^{-1}df$) with the group of based gauge transformation. In this case H is the moduli space of gauge connections. In the same way, if $N = Map_0(S^n, K)$ is the group of based gauge transformation of a (trivial) K-bundle over S^n, then the moduli space of gauge connections on S^n is, up to a homotopy, equal to the group $H = Map_0(S^{n-1}, K)$. These cases become interesting when n is odd since there are nontrivial abelian extensions of N described by an element in $\mathrm{H}^2_s(N, A)$, where A is the abelian group of circle valued functions on $G = PH$. In quantum field theory elements of H^2_s describe Hamiltonian anomalies, due to chiral symmetry breaking in the quantization of Weyl fermions [2], Chapters 4 and 5, and [3] for an index theory derivation.

The 3-cocycles on H are also relevant in the categorical representation theory. Consider a category $\mathcal{C}$ of irreducible representations of some associative algebra $\mathcal{B}$ in a complex vector space V, the morphisms being isomorphisms of representations. Suppose that for each $h \in H$ there is a given functor $F_h : \mathcal{C} \to \mathcal{C}$ and for each pair $h, k \in H$ an isomorphism $i_{h,k} : F_h \circ F_k \to F_{hk}$. Then both $i_{j,hk} \circ i_{h,k}$ and $i_{jh,k} \circ i_{j,h}$ are isomorphisms $F_j \circ F_h \circ F_k \to F_{jhk}$ but not necessarily equal;

$$i_{j,hk} \circ i_{h,k} = \alpha(j, h, k) i_{jh,k} \circ i_{j,h}$$

for some $\alpha(j, h, k)$ in $\mathbb{C}^\times$. The function α is then a $\mathbb{C}^\times$-valued cocycle on H.

A simple example of the above functorial representation is the case studied in [4]. There the algebra $\mathcal{B}$ is the algebra of canonical anticommutation relations (CAR) generated by vectors in the space $V = L^2(S^1, \mathbb{C}^n)$ of square-integrable vector valued functions on the unit circle and the category of representations are special quasi-free representations of $\mathcal{B}$ which are parametrized by polarizations $V = V_+(a) \oplus V_-(a)$, to positive/negative modes of the Dirac operator on the circle coupled to the gauge connection $a = f^{-1}df$ for $f \in PH$. (One must choose boundary conditions for $f : [0, 2\pi] \to H$ such that a is smooth and periodic.) The group H acts by $n \times n$ unitary matrices on $\mathbb{C}^n$ and the group $G = PH$ acts by point-wise multiplication on functions. Fix a (locally smooth) section $\psi : H \to G$. Then each $h \in H$ defines an automorphism of the algebra $\mathcal{B}$ through the action of $\psi(h)$ in V. The quasi-free representations of $\mathcal{B}$ are defined in a Fock space $\mathcal{F}$. Given a pair $h, k \in H$ the element $\ell_{h,k} := \psi(h)\psi(k)\psi(hk)^{-1}$ is contained in $N = \Omega G \subset G$ and choosing an element $i_{h,k} \in \hat{N}$ in the fiber above $\ell_{h,k}$ one obtains an automorphism of a quasi-free representation parametrized by a. The functor F_h for $h \in H$ takes the quasi-free representation corresponding to any $\psi(k)$ to the quasi-free representation parametrized by $\psi(kh)$. The role of Lie algebra 3-cocycles in C^* algebraic constructions in quantum field theory was also studied earlier in [5].

The article is structured as follows. After this introduction and a preliminary section on infinite-dimensional Lie theory, we provide in Section 3 conditions which assure that certain automorphisms defined on the building blocks of an abelian Lie group extension can be lifted to the extension itself. Next, let G be an (infinite-dimensional) Lie group, N a normal subgroup of G, $H = G/N$ and A a G-module. Moreover, let $\widehat{N}$ be an extension of N by A (viewed as an abelian group) described by a representative of an element $c_2 \in \mathrm{H}^2_s(N, A)$, i.e., by a locally smooth 2-cocycle with values in A. The aim of Section 4 is to find a transgression $\tau(c_2) \in \mathrm{H}^3_s(H, A^N)$, where A^N is the set of N invariant elements in A. This we achieve by using methods from the theory of smooth crossed modules and under the assumption that the cohomology group $\mathrm{H}^1_s(N, A)$ vanishes (cf. Theorem 4.6). In addition, under the same restriction, we show that the transgressed element $\tau(c_2)$ vanishes if and only if there is a prolongation of the group extension $1 \to N \to G \to H \to 1$ to an extension $1 \to \widehat{N} \to \widehat{G} \to H \to 1$ (cf. Theorem 4.7). In Section 5 we shall give a different construction of lifting the automorphisms of N to automorphisms of an abelian extension $\widehat{N}$ in the case when A is the G-module of smooth maps from G to the unit circle S^1. This case is important in the gauge theory applications. Section 6 presents some results on the transgression $\mathrm{H}^2_s(N, A) \to \mathrm{H}^3_s(H, A^N)$ in the case of the module $A = Map(G, S^1)$, while Section 6 is devoted to the case of a non simply connected base H, e.g., a torus and, further, to a gauge theory application. The goal of the first part of the Appendix is to extend the geometric construction of the central extension $\widehat{LG}$ of the smooth loop group LG, with G a simply connected compact Lie group, [6], to the case when G is a connected compact Lie group but not necessarily simply connected. In the second part of the Appendix we discuss a Fock space construction of gerbes over G/Z. Finally, in the third part of the Appendix we show that a certain second cohomology group acts in a natural way as a simply transitive transformation group on the set of equivalence classes of extensions of a given smooth crossed module, which is related to our results in Section 4.

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