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On the inversion of Fueter's theorem

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ABSTRACT

The well known Fueter theorem allows to construct quaternionic regular functions or monogenic functions with values in a Clifford algebra defined on open sets of Euclidean space $\mathbb{R}^{n+1}$, starting from a holomorphic function in one complex variable or, more in general, from a slice hyperholomorphic function. Recently, the inversion of this theorem has been obtained for odd values of the dimension n. The present work extends the result to all dimensions n by using the Fourier multiplier method. More precisely, we show that for any axially monogenic function f defined in a suitable open set in $\mathbb{R}^{n+1}$, where n is a positive integer, we can find a slice hyperholomorphic function $\vec{f}$ such that $f = \Delta^{(n-1)/2}\vec{f}$. Both the even and the odd dimensions are treated with the same, viz., the Fourier multiplier, method. For the odd dimensional cases the result obtained by the Fourier multiplier method coincides with the existing result obtained through the pointwise differential method.

1. Introduction

The Fueter theorem is a useful tool to generate Cauchy–Fueter regular functions from holomorphic functions in the upper half complex plane $\mathbb{C}^+$, see [1]. Furthermore, let O be an open subset of $\mathbb{C}^+$, $f(z) = u(s,t) + iv(s,t)$ be a holomorphic function defined on O and $\mathbb{H}$ be the set of all quaternions. Ω_q is an open subset of $\mathbb{H}$ and is induced by O, i.e., $\Omega_q = \{q = q_0 + \underline{q} \in \mathbb{H} \mid (q_0, |\underline{q}|) \in O\}$, where $q_0, \underline{q} := q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ denote the real and the imaginary part of the quaternion q, respectively. In Ω_q, the function

$$F(q_0, \underline{q}) := \Delta_q \left(u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(q_0, |\underline{q}|) \right)$$

is both left and right regular (or monogenic) with respect to the quaternionic Cauchy–Riemann operator

$$D_q = \partial_{q_0} + \mathbf{i}\partial_{q_1} + \mathbf{j}\partial_{q_2} + \mathbf{k}\partial_{q_3},$$

i.e., F satisfies $D_qF = FD_q = 0$, where $\Delta_q = \partial^2_{q_0} + \partial^2_{q_1} + \partial^2_{q_2} + \partial^2_{q_3}$ denotes the Laplacian operator in the four real variables $q_0, \ldots, q_3$.

Qian by means of Fueter's theorem developed a singular integral theory on the quaternionic unit sphere and its Lipschitz perturbations that corresponds to the operator algebra of the spherical Dirac operator, which is identical with the

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Cauchy–Dunford H^{∞} functional calculus of the spherical Dirac operator. He also developed a theory of bounded holomorphic Fourier multipliers in the Coifman–Meyer formulation of Fourier transformation on the Lipschitz surfaces [2].

Under the above assumptions on f, in 1957, Fueter's theorem was extended to $\mathbb{R}^{n+1}$ by Sce [3] for odd values of the dimension n. Specifically, taken a function f as above, the Clifford algebra valued function

$$G(x_0, \underline{x}) := \Delta^{\frac{n-1}{2}} \left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right), \quad x = x_0 + \underline{x} \in \mathbb{R}^{n+1}, \tag{1.1}$$

is left and right monogenic with respect to the generalized Cauchy–Riemann operator in $\mathbb{R}^{n+1}$

$$D := \partial_{x_0} + \sum_{i=1}^{n} \mathbf{e}_i \partial_{x_i}, \tag{1.2}$$

i.e., $DG = GD = 0$, where

$$\Delta := \sum_{i=0}^{n} \partial_{x_i}^2 \tag{1.3}$$

is the Laplacian in the $n + 1$ real variables. Moreover, the function G as in (1.1) turns out to be axially monogenic, namely it is monogenic and it has the form $A(x_0, |\underline{x}|) + (\underline{x}/|\underline{x}|)B(x_0, |\underline{x}|)$ where A and B are real-valued (more in general, for an axially monogenic function A and B may have values in a Clifford algebra).

Qian in 1997 extended Sce's result to $\mathbb{R}^{n+1}$ for all positive integers n. In fact the Fourier multiplier method used by Qian for n even is also valid for n odd, see [4,5] and also [6]. To the author's knowledge, the approach of using Fueter's theorem is, so far, unique, in establishing the singular integral operator algebra theory on the sphere and its Lipschitz perturbations. In contrast, the analogous theories for various contexts of unbounded Lipschitz graphs of one and higher dimensions were established with a considerable variety of methods [7–11].

Fueter theorem can also be understood in terms of representation theory as an intertwining map between some sl(2)-modules, see [12]. In particular, it is related with some properties of Gegenbauer polynomials. These polynomials are important in the representation theory for the spin group Spin(n), within the setting of branching rules and axially monogenic polynomials on $\mathbb{R}^n$. Fueter theorem is also connected with some Appell sequences, see [13].

For further generalizations of Fueter's Theorem beyond Sce and Qian we refer the reader to [14–19].

It is natural to ask whether there exists any converse result: given a monogenic function, is it possible to find its Fueter's primitive? The main goal of this paper is to show that the Fueter mapping is surjective on the set of axially monogenic functions, and it is possible to solve the following inverse problem for all dimensions n:

Problem 1.1. Given an axially monogenic function

$$f(x) = A(x_0, r) + \underline{\omega} B(x_0, r),$$

where $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, $r := |\underline{x}|$, $\underline{\omega} := \underline{x}/r$, and $A(x_0, r)$, $B(x_0, r)$ are Clifford algebra valued functions, determine a slice hyperholomorphic function $\vec{f}$ (the so-called Fueter's primitive) such that

$$f(x) = \Delta^{\frac{n-1}{2}} \vec{f}(x).$$

We note that for n odd integer this result was previously proved by Colombo et al. in [20,21] in which $\Delta^{(n-1)/2}$ is a pointwise differential operator. In this paper, we give a uniform treatment for all integers n in which $\Delta^{(n-1)/2}$ is defined by the corresponding Fourier multiplier and show that when n is odd our result coincides with the above mentioned result in [21].

The proof is a combination of the method used by Colombo, Sabadini, Sommen in [21], as main strategy, and Qian's Fourier multiplier method as given in [4,5], as technical approach. Although the Fourier multiplier method is applicable only to the Clifford Cauchy kernel type meromorphic functions, through the Clifford Cauchy integral formula for general domains one can obtain Fueter's inversion for a general class of axially monogenic functions.

It is worthwhile mentioning that the method used in [21] can be further generalized to obtain Fueter's inversion for axially monogenic functions of degree k, see [22], and then obtain results in the case of monogenic functions, when the dimension n is odd. Using a completely different approach, it is also possible to relate the class of slice hyperholomorphic functions to the class of monogenic functions by using the Radon transform and its dual, see [23]. This approach does not depend on the dimension n considered.

The plan of the paper is organized as follows. Section 2 contains preliminary results on Clifford algebras, monogenic functions, Fueter's theorem and its inversion. In Section 3 we recall some basic facts on slice hyperholomorphic functions. Section 4 is the core of the paper and is devoted to obtain the inversion of Fueter's theorem.