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A bicategory of reduced orbifolds from the point of view of differential geometry

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ABSTRACT

We describe a bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ of reduced orbifolds in the framework of classical differential geometry (i.e. without any explicit reference to the notions of Lie groupoids or differentiable stacks, but only using orbifold atlases, local lifts and changes of charts). In order to construct such a bicategory, we firstly define a 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ whose objects are reduced orbifold atlases (on any paracompact, second countable, Hausdorff topological space). The definition of morphisms is obtained as a slight modification of a definition by A. Pohl, while the definitions of 2-morphisms and compositions of them are new in this setup. Using the bicalculus of fractions described by D. Pronk, we are able to construct the bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ from the 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. We prove that $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ is equivalent to the bicategory of reduced orbifolds described in terms of proper, effective, étale Lie groupoids by D. Pronk and I. Moerdijk and to the well-known 2-category of reduced orbifolds constructed from a suitable class of differentiable Deligne–Mumford stacks.

0. Introduction

A well-known issue in mathematics is that of modeling geometric objects where points have non-trivial groups of automorphisms. In topology and differential geometry the standard approach to these objects (when each point has a finite group of automorphisms) is through orbifolds. This concept was formalized for the first time by Ikiro Satake in 1956 in [1] with some different hypotheses than the current ones, although the informal idea dates back at least to Henri Poincaré (for example, see [2]). Currently there are at least 3 main approaches to orbifolds:

(1) via orbifold atlases and "good maps" between them, as described in [3],
(2) via the 2-category of proper, étale (Lie) groupoids, "localized" with respect to weak equivalences (see for example [4,5] and [6]),
(3) via a family of C^∞-Deligne–Mumford stacks (see for example [7] and [8]).

On the one hand, the approach in (1) gives rise to a 1-category. On the other hand, the approach in (2) gives rise to a bicategory (i.e. almost a 2-category, where compositions of 1-morphisms is associative only up to canonical 2-morphisms) and the approach in (3) gives rise to a 2-category. It was proved in [4] that (2) and (3) are equivalent as bicategories. Since (2) and (3) are compatible approaches, then one might argue that:

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(i) there should also exist a non-trivial structure of 2-category or bicategory, having as objects orbifold atlases or equivalence classes of them (i.e. orbifold structures);
(ii) the structure of (i) should be compatible with the approaches of (2) and (3), and it should replace the approach of (1) (since (1) gives rise only to a 1-category instead of a 2-category or bicategory).

In the present paper we will manage to prove both (i) and (ii) for the family of all *reduced* orbifolds, i.e. orbifolds that are locally modeled on open connected sets of some $\mathbb{R}^n$, modulo finite groups acting smoothly and *effectively* on them. In order to do that, we proceed as follows.

- We describe a 2-category $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ whose objects are reduced orbifold atlases on any paracompact, second countable, Hausdorff topological space. The definition of morphisms is obtained as a slight modification of an analogous definition given by Anke Pohl in [9], while the notion of 2-morphisms (and compositions of them) is new in this setup (see Definition 1.9). Such notions are useful for differential geometers mainly because they do not require any previous knowledge of Lie groupoids and/or differentiable stacks. $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ is a 2-category, but it is still not the structure that we want to get in (i); indeed in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ different orbifold atlases representing the same orbifold structure in general are not related by an isomorphism or by an internal equivalence.
- We recall briefly the definition of the 2-category $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$, whose objects are proper, effective, étale differentiable groupoids, and we describe in Theorem 3.15 a 2-functor $\mathcal{F}^{\,\mathrm{red}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \to (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$.
- In [4] Dorette Pronk proved that the class $\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}$ of all weak equivalences in $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})$ (also known as essential equivalences) admits a right bicalculus of fractions. Roughly speaking, this amounts to saying that there are a bicategory $(\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$ and a pseudofunctor
$$\mathcal{U}_{\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}} : (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}) \longrightarrow (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right]$$
that sends each weak equivalence to an internal equivalence, and that is universal with respect to this property. The bicategory obtained in this way is the bicategory that we mentioned in (2) above, if we restrict to the case of reduced orbifolds.
- In $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$ we select a class $\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}$ of morphisms (that we call "refinements" of reduced orbifold atlases, see Definition 5.1), and we prove that such a class admits a right bicalculus of fractions. Therefore, we are able to construct a bicategory $(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ and a pseudofunctor
$$\mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}} : (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) \longrightarrow (\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) := (\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})\left[\mathbf{W}^{-1}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}\right]$$
that sends each refinement to an internal equivalence, and that is universal with respect to this property (see Proposition 6.1). Objects in this new bicategory are again reduced orbifold atlases; a morphism from an atlas $\mathcal{X}$ to an atlas $\mathcal{Y}$ is any triple consisting of a reduced orbifold atlas $\mathcal{X}'$, a refinement $\mathcal{X}' \to \mathcal{X}$ and a morphism $\mathcal{X}' \to \mathcal{Y}$. In other terms, a morphism from $\mathcal{X}$ to $\mathcal{Y}$ is given firstly by replacing $\mathcal{X}$ with a "refined" atlas $\mathcal{X}'$ (keeping track of the refinement), then by considering a morphism from $\mathcal{X}'$ to $\mathcal{Y}$ in $(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl})$. We refer to Description 6.3 for the notion of 2-morphisms in this bicategory.
- Lastly, using the results about bicategories of fractions that we proved in our previous papers [10] and [11], we are able to prove that:

Theorem A (Proposition 7.5 *and* Theorem 8.3)**.** *There is a pseudofunctor $\mathcal{G}^{\,\mathrm{red}}$ (explicitly constructed), making the next diagram commute; assuming the axiom of choice, $\mathcal{G}^{\,\mathrm{red}}$ is an equivalence of bicategories.*

$$\begin{array}{ccc}
(\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}) & \xrightarrow{\quad\mathcal{F}^{\,\mathrm{red}}\quad} & (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}) \\
{\scriptstyle \mathcal{U}_{\mathbf{W}_{\mathcal{R}\mathbf{ed}\,\mathcal{A}\mathbf{tl}}}}\big\downarrow & \curvearrowright & \big\downarrow{\scriptstyle \mathcal{U}_{\mathbf{W}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}}} \\
(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb}) & \dashrightarrow_{\mathcal{G}^{\,\mathrm{red}}} & (\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd})\left[\mathbf{W}^{-1}_{\mathcal{PE\acute{E}}\,\mathcal{G}\mathbf{pd}}\right].
\end{array} \qquad (0.1)$$

Since (2) and (3) are equivalent approaches by [4], this implies at once that:

Theorem B (Theorem 8.4)**.** *$(\mathcal{R}\mathbf{ed}\,\mathcal{O}\mathbf{rb})$ is equivalent to the 2-category $(\mathbf{Orb}^{\mathbf{eff}})$ of effective orbifolds described as a full 2-subcategory of the 2-category of C^∞-Deligne–Mumford stacks.*

In all this paper we will not use explicitly the language of stacks; however, it is important to remark that:

- in the language of (differentiable) stacks, the notion of objects is complicated and does not provide a simple geometric intuition, since it is based on the notions of pseudofunctor (or category fibered in groupoids), Grothendieck topology and descent conditions. However, having managed to understand stacks, 1-morphisms and 2-morphisms are almost straightforward to define and the resulting structure is that of a 2-category;