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Twisted partially pure spinors

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ABSTRACT

Motivated by the relationship between orthogonal complex structures and pure spinors, we define twisted partially pure spinors in order to characterize spinorially subspaces of Euclidean space endowed with a complex structure.

1. Introduction

Spinors have played an important role in both physics and mathematics ever since they were discovered by É. Cartan in 1913. Cartan defined *pure spinors* [1–3] in order to characterize (almost) complex structures. Pure spinors have been present implicitly in both the Penrose formalism of General Relativity within the notion of "flag planes" [4,5] and, more recently, in Seiberg–Witten theory, since every non-zero positive spinor is pure in 4 dimensions [6].

The notion of abstract CR structure in odd dimensions generalizes that of complex structure in even dimensions, and aims to describe intrinsically the property of being a hypersurface of a complex space form. It has been proved that every strictly pseudoconvex CR manifold has a canonical $Spin^c$ structure [7]. This fact, and the relation of pure spinors to complex structures, naturally led us to ask whether it is possible to characterize CR structures by the existence of special spinor fields.

In this paper, we set up the algebraic preliminaries for such an endeavour. More precisely, we characterize subspaces of Euclidean space $\mathbb{R}^n$ endowed with an orthogonal complex structure by means of twisted spinors, which is a generalization of the relation between classical pure spinors and orthogonal complex structures on Euclidean space $\mathbb{R}^{2m}$. Recall that a classical pure spinor $\phi \in \Delta_{2m}$ is a spinor such that the (isotropic) subspace of complexified vectors $X - iY \in \mathbb{R}^{2m} \otimes \mathbb{C}$, $X, Y \in \mathbb{R}^{2m}$, which annihilate ϕ under Clifford multiplication, denoted by "$\cdot$",

$$(X - iY) \cdot \phi = 0$$

is of maximal dimension, where $m \in \mathbb{N}$ and Δ_{2m} is the standard complex representation of the Spin group $Spin(2m)$ (cf. [8]). This means that for every $X \in \mathbb{R}^{2m}$ there exists a $Y \in \mathbb{R}^{2m}$ satisfying

$$X \cdot \phi = iY \cdot \phi.$$

By setting $Y = J(X)$, one can see that a pure spinor determines a complex structure on $\mathbb{R}^{2m}$. Geometrically, the two subspaces $TM \cdot \phi$ and $iTM \cdot \phi$ of Δ_{2m} coincide, which means $TM \cdot \phi$ is a complex subspace of Δ_{2m}, and the effect of multiplication by the number $i = \sqrt{-1}$ is transferred to the tangent space TM in the form of J.

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The authors of [9,10] investigated (the classification of) non-pure classical spinors by means of their isotropic subspaces. In [10], the authors noted that there may be many spinors (belonging to different orbits under the action of the Spin group) admitting isotropic subspaces of the same dimension, and that there is a gap in the possible dimensions of such isotropic subspaces. In our Euclidean/Riemannian context, such isotropic subspaces correspond to subspaces of Euclidean space endowed with orthogonal complex structures. Here, we define twisted partially pure spinors (cf. Definition 3.1) in order to establish a one-to-one correspondence between subspaces of Euclidean space (of a fixed codimension) endowed with orthogonal complex structures (and oriented orthogonal complements), and orbits of such spinors under a particular subgroup of the twisted Spin group (cf. Theorem 3.1). By using spinorial twists we avoid having different orbits under the full twisted Spin group and also the aforementioned gap in the dimensions.

As we mentioned earlier, the need to establish such a correspondence arises from our interest in developing a spinorial setup to study the geometry of manifolds admitting (almost) CR structures (of arbitrary codimension) and elliptic structures. Since such manifolds are not necessarily Spin nor Spin^c, we are led to consider spinorially twisted Spin groups, representations, structures, etc. Geometric and topological considerations regarding such manifolds will be presented in [11].

The paper is organized as follows. In Section 2 we recall basic material on Clifford algebras, Spin groups and representations; we define the twisted Spin groups and representations that will be used, the space of anti-symmetric 2-forms and endomorphisms associated to twisted spinors; we also present some results on subgroups and branching of representations. In Section 3, we define partially pure spinors, deduce their basic properties and prove the main theorem, Theorem 3.1, which establishes the aforementioned one-to-one correspondence.

2. Preliminaries

In this section, we briefly recall basic facts about Clifford algebras, the Spin group and the standard Spin representation [12]. We also define the twisted Spin groups and representations, the antisymmetric 2-forms and endomorphisms associated to a twisted spinor, and describe various inclusions of groups into (twisted) Spin groups.

2.1. Clifford algebras

Let Cl_n denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j \cdot e_k + e_k \cdot e_j = -2\left\langle e_j, e_k\right\rangle,$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^n$. The even Clifford subalgebra Cl_n^0 is defined as the invariant $(+1)$-subspace of the involution of Cl_n induced by the map $-\mathrm{Id}_{\mathbb{R}^n}$. Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$$

denote the complexification of Cl_n. The Clifford algebras are isomorphic to matrix algebras. In particular,

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\ \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k+1, \end{cases}$$

where

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{k \text{ times}}$$

is the tensor product of $k = [\frac{n}{2}]$ copies of $\mathbb{C}^2$. The map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\mathbb{C}^{2^k})$$

is defined to be either the above mentioned isomorphism if n is even, or the isomorphism followed by the projection onto the first summand if n is odd. In order to make κ explicit, consider the following matrices

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators $e_1, \ldots, e_n$, κ can be described explicitly as follows,

$$\begin{aligned}
e_1 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes Id \otimes g_1, \\
e_2 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes Id \otimes g_2, \\
e_3 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes g_1 \otimes T, \\
e_4 &\mapsto Id \otimes Id \otimes \cdots \otimes Id \otimes g_2 \otimes T, \\
&\vdots \qquad \ldots \\
e_{2k-1} &\mapsto g_1 \otimes T \otimes \cdots \otimes T \otimes T \otimes T, \\
e_{2k} &\mapsto g_2 \otimes T \otimes \cdots \otimes T \otimes T \otimes T,
\end{aligned}$$