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Quasi-periodic solutions to the hierarchy of four-component Toda lattices

Jiao Wei *, Xianguo Geng, Xin Zeng

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, People's Republic of China

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ABSTRACT

Starting from a discrete 3×3 matrix spectral problem, the hierarchy of four-component Toda lattices is derived by using the stationary discrete zero-curvature equation. Resorting to the characteristic polynomial of the Lax matrix for the hierarchy, we introduce a trigonal curve $\mathcal{K}_{m-2}$ of genus $m-2$ and present the related Baker–Akhiezer function and meromorphic function on it. Asymptotic expansions for the Baker–Akhiezer function and meromorphic function are given near three infinite points on the trigonal curve, from which explicit quasi-periodic solutions for the hierarchy of four-component Toda lattices are obtained in terms of the Riemann theta function.

1. Introduction

The Toda lattice

$$x_{n,tt} = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1}), \quad (n, t) \in \mathbb{Z} \times \mathbb{R} \tag{1.1}$$

for sequence $x_n = x(n, t)$, is a typical differential–difference system in soliton theory, which was originally derived in 1966 as a model for waves in lattices composed of particles interacting by nonlinear (exponential) forces between nearest neighbours [1]. In terms of the variable transformation $u_n = -\exp(x_n - x_{n+1})$, $v_n = x_{n,t}$, it can be rewritten in the following form

$$\begin{aligned} u_{n,t} &= u_n(v_n - v_{n+1}), \\ v_{n,t} &= u_n - u_{n-1}. \end{aligned} \tag{1.2}$$

Since then, the study of the lattice soliton equations has received considerable attention [2–5]. The main reason is that a number of physical phenomena can be modelled by nonlinear lattice equations. Moreover, some generalizations of the Toda lattice equations have been discussed in a series of papers [6–11].

The main aim of the present paper is to study the hierarchy of four-component Toda lattices associated with a discrete 3×3 matrix spectral problem and the corresponding trigonal curve with three infinite points on the basis of the approaches in Refs. [12–16], from which explicit quasi-periodic solutions for the hierarchy of four-component Toda lattices are obtained in terms of the Riemann theta function. The first nontrivial member in the hierarchy is the four-component Toda lattice [11]

$$\begin{aligned} u_{n,t} &= u_n(w_{n+1}r_{n+1} - w_n r_n - v_{n+1} + v_n), \\ v_{n,t} &= u_n - u_{n-1}, \\ w_{n,t} &= -u_n w_{n+1}, \\ r_{n,t} &= u_{n-1} r_{n-1}, \end{aligned} \tag{1.3}$$

* Corresponding author.
E-mail addresses: weijiaozzu@sohu.com (J. Wei), xggeng@zzu.edu.cn (X. Geng), xzeng@zzu.edu.cn (X. Zeng).

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which is exactly the Toda lattice system (1.2) as $w_n = r_n = 0$. Before turning to the contents of each section, it is necessary to review the related literature on the subject. Quasi-periodic solutions for a lot of soliton equations associated with 2×2 matrix spectral problems have been obtained such as the KdV, Toda lattice, nonlinear Schrödinger equations and others [17–30]. However, when our sight turns to the 3×3 matrix spectral problems, the research becomes more difficult and complicated because of concerning the theory of trigonal curves rather than the hyperelliptic curves in the 2×2 cases [31–39]. In spite of this, a unified framework was proposed which yields all quasi-periodic solutions of the entire Boussinesq hierarchy associated with the third-order differential operator [12,13]. Recently, based on the characteristic polynomial of a Lax matrix associated with the 3×3 matrix spectral problem, a general method was developed to introduce the trigonal curve, from which the unified framework was successfully generalized to obtain the quasi-periodic solutions for the modified Boussinesq, the Kaup–Kupershmidt, the coupled mKdV hierarchies and others associated with the continuous 3×3 matrix spectral problems [14–16,40,41].

The present paper is arranged as follows. In the next section, with the aid of the discrete zero-curvature equation and three sets of Lenard recursion equations, we derive the hierarchy of four-component Toda lattices associated with the discrete 3×3 matrix spectral problem. In Section 3, we introduce the Baker–Akhiezer function and a trigonal curve of $\mathcal{K}_{m-2}$ of degree $m-2$ with the help of the characteristic polynomial of Lax matrix for the hierarchy. The meromorphic function closely related to the Baker–Akhiezer function is defined on $\mathcal{K}_{m-2}$. Then the lattice hierarchy is decomposed into a system of Dubrovin-type ordinary differential equations. In Section 4, we derive the asymptotic expansions of the meromorphic function near infinite points with the help of the Riccati-type equation and obtain the divisor accordingly. Then the asymptotic properties of the Baker–Akhiezer function are also studied. The last section is devoted to present the quasi-periodic solutions of the meromorphic function ϕ, the Baker–Akhiezer function ψ_1 and solutions for the hierarchy of four-component Toda lattices. Moreover, the continuous and discrete flows for the hierarchy of four-component Toda lattices are straightened out by virtue of a meromorphic differential.

2. The hierarchy of four-component Toda lattices

Throughout this paper we suppose the following hypothesis. *Assume that* u, v, w, r *satisfy* $u(\cdot,t), v(\cdot,t), w(\cdot,t), r(\cdot,t) \in \mathbb{C}^{\mathbb{Z}}$, $t \in \mathbb{R}$, $u(n,\cdot), v(n,\cdot), w(n,\cdot), r(n,\cdot) \in C^1(\mathbb{R})$, $n \in \mathbb{Z}$, *where* $\mathbb{C}^{\mathbb{Z}}$ *denotes the set of all complex-valued sequences indexed by* $\mathbb{Z}$.

Let us define the shift operators and difference operator by

$$Ef(n) = f(n+1), \qquad E^{-1}f(n) = f(n-1), \qquad \Delta f(n) = (E-1)f(n), \quad n \in \mathbb{Z}.$$

For the sake of convenience, we usually use the notations $f(n) = f, f(n+k) = E^k f, E^{\pm} f = f^{\pm}, n, k \in \mathbb{Z}$. Consider the discrete 3×3 matrix spectral problem [11]

$$E\psi = U\psi, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & u & 0 \\ 1 & \lambda + v & r \\ 0 & w & 1 \end{pmatrix}, \tag{2.1}$$

where u, v, w, r are four potentials and λ is a constant spectral parameter. In order to derive the hierarchy of lattice equations associated with (2.1), we first solve the stationary discrete zero-curvature equation:

$$(EV)U - UV = 0, \qquad V = (V_{ij})_{3\times 3}, \tag{2.2}$$

which is equivalent to

$$\begin{aligned}
&V_{12}^+ - uV_{21} = 0,\\
&uV_{11}^+ + (\lambda+v)V_{12}^+ + wV_{13}^+ - uV_{22} = 0,\\
&rV_{12}^+ + V_{13}^+ - uV_{23} = 0,\\
&V_{22}^+ - V_{11} - (\lambda+v)V_{21} - rV_{31} = 0,\\
&uV_{21}^+ + (\lambda+v)V_{22}^+ + wV_{23}^+ - V_{12} - (\lambda+v)V_{22} - rV_{32} = 0,\\
&rV_{22}^+ + V_{23}^+ - V_{13} - (\lambda+v)V_{23} - rV_{33} = 0,\\
&V_{32}^+ - wV_{21} - V_{31} = 0,\\
&uV_{31}^+ + (\lambda+v)V_{32}^+ + wV_{33}^+ - wV_{22} - V_{32} = 0,\\
&rV_{32}^+ + V_{33}^+ - wV_{23} - V_{33} = 0,
\end{aligned} \tag{2.3}$$

where each entry $V_{ij} = V_{ij}(a,b,c,d,e)$ is a Laurent expansion in λ:

$$\begin{aligned}
&V_{11} = -(\lambda+v)b + c^+ - rd, && V_{12} = u^-b^-, && V_{13} = u^-a^-,\\
&V_{21} = b, && V_{22} = c, && V_{23} = a + rb,\\
&V_{31} = d, && V_{32} = w^-b^- + d^-, && V_{33} = e,\\
&a = \sum_{j\geq 0} a_j\lambda^{-j}, \quad b = \sum_{j\geq 0} b_j\lambda^{-j}, && c = \sum_{j\geq 0} c_j\lambda^{-j}, \quad d = \sum_{j\geq 0} d_j\lambda^{-j}, && e = \sum_{j\geq 0} e_j\lambda^{-j}.
\end{aligned} \tag{2.4}$$