Journal of Geometry and Physics 106 (2016) 42–50

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Equilibria of three constrained point charges

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ARTICLE INFO

Article history:
Received 15 February 2016
Accepted 4 March 2016
Available online 14 March 2016

Keywords:
Coulomb energy
Pitchfork bifurcation
Euler's buckling beam
Morse points

ABSTRACT

We study the critical points of Coulomb energy considered as a function on configuration spaces associated with certain geometric constraints. Two settings of such kind are discussed in some detail. The first setting arises by considering polygons of fixed perimeter with freely sliding positively charged vertices. The second one is concerned with triples of positive charges constrained to three concentric circles. In each of these cases the Coulomb energy is generically a Morse function. We describe the minima and other stationary points of Coulomb energy and show that, for three charges, a pitchfork bifurcation takes place accompanied by an effect of the Euler's Buckling Beam type.

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1. Introduction

We deal with equilibrium configurations of point charges with Coulomb interaction satisfying certain geometric constraints. Our approach to this topic is similar to the paradigms used in [1,2]. Namely, we consider the Coulomb energy as a function on a certain configuration space naturally associated with the constraints in question, and investigate its critical points. The main attention in this paper is given to two specific problems naturally arising in this setting. The first one deals with identification and calculation of equilibrium configurations of charges satisfying the given constraints. The second one, called the *inverse problem*, is concerned with characterizing those configurations of points for which there exists a collection of non-zero charges such that the given configuration is a critical point of Coulomb energy on the corresponding configuration space. Such configurations are called *Coulomb equilibria*.

The geometric constraints considered below come from two sources: (i) Coulomb energy of point charges freely sliding along a flexible planar contour of fixed length, and (ii) Coulomb energy of concentric orbitally constrained triples of charges. The first setting was inspired by the concept of "necklace with interacting beads" introduced and investigated by P. Exner [3]. This setting has been considered in a similar situation in our previous paper [4].

It turns out that interesting results exist even in the case of three charges. We focus on the minimum energy and the other stationary points and values. While for almost equal charges the minimum is achieved on a triangle configuration, it turns out that in both settings the global minimum is achieved in an aligned situation if one of the charges is much smaller (or bigger, depending on the setting) than the others. The transition between these two states exhibits the well-known *supercritical pitchfork bifurcation* accompanied by a *fixing* effect, similar to the Euler Buckling Beam phenomenon [5,6].

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http://dx.doi.org/10.1016/j.geomphys.2016.03.006
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It seems worthy of adding that in most of our considerations the Coulomb forces can be replaced by various other central forces. The qualitative behavior will be the same with modifications on the quantitative side.

2. Configuration spaces and Coulomb energy

We are basically interested in studying the equilibrium configurations of a system of repelling charges. As usual equilibria are defined as the critical (stationary) points of Coulomb energy of an n-tuple of points $\{p_i\}$ defined by the formula

$$E = \sum_{i<j} \frac{q_i q_j}{d_{ij}},$$

where q_i are some positive numbers (charges) assigned to the points p_i, and d_{ij} are the distances $|p_i p_j|$. To this end the Coulomb energy is considered as a function on a certain configuration space naturally associated with the given geometric constraints.

Our first configuration space is the space of all labeled n-tuples of points[1] with the constraint that the perimeter is not bigger than 1:

$$\overline{M}(n) = \left\{ (p_1, \ldots, p_n) \mid p_i \in \mathbb{R}^2, p_1 = 0, \sum_{i=1}^{n} |p_i p_{i+1}| \leq 1 \right\} \Big/ SO(2).$$

Informally, one can think of a closed rope with freely sliding positively charged points on it. Factorization by $SO(2)$ means that we are only interested in the shape of a configuration and ignore orientation preserving motions. However, we do not identify symmetric n-tuples.

Let us also introduce the space

$$M(n) = \left\{ (p_1, \ldots, p_n) \mid p_i \in \mathbb{R}^2, p_1 = 0, \sum_{i=1}^{n} |p_i p_{i+1}| = 1 \right\} \Big/ SO(2).$$

It is naturally identified with the space of all planar polygons with fixed perimeter (vertices are allowed to coincide) factorized by orientation preserving motions. The elements of the space $M(n)$ are called either *polygons* or *configurations.*

For our purposes, it is important to know the topological type of the configuration space.

Theorem 1. *The space $M(n)$ is diffeomorphic to the complex projective space $\mathbb{C}P^{n-2}$.*

Proof. By definition,

$$\mathbb{C}P^{n-2} = \{(u_1 : \ldots : u_{n-1}) \mid u_i \in \mathbb{C}, \text{ not all } u_i = 0\},$$

assuming that two proportional $(n-1)$-tuples are identified. We add one more term and write

$$\mathbb{C}P^{n-2} = \left\{ \left(u_1 : \ldots : u_{n-1} : -\sum_{i=1}^{n-1} u_i \right) \middle| \, u_i \in \mathbb{C}, \text{ not all } u_i = 0 \right\},$$

with identification

$$\left(u_1 : \ldots : u_{n-1} : -\sum_{i=1}^{n-1} u_i \right) = \left(\lambda \cdot u_1 : \ldots : \lambda \cdot u_{n-1} : -\lambda \cdot \sum_{i=1}^{n-1} u_i \right).$$

This can be interpreted as the space of all n-gons with non-zero perimeter. Indeed, complex numbers u_i yield vectors in the plane. The factorization by multiplication by complex numbers amounts to factorization of the space of polygons by all possible rotations and scalings. □

The space $\overline{M}(n)$ is a cone over $M(n)$, and we have a natural inclusion $\overline{M}(n) \supset M(n)$. All the polygons with non-zero perimeter strictly smaller than 1 form a manifold diffeomorphic to $\mathbb{C}P^{n-2} \times \mathbb{R}$, so it makes sense to speak of critical points of the Coulomb energy E. The informal message of the following proposition is that the "sliding charges on a closed rope" problem reduces to "fixed perimeter" problem.

Proposition 1. *The Coulomb energy has no critical points in $\overline{M}(n)$ outside $M(n)$.*

Proof. Assume P is a critical polygon whose perimeter is strictly smaller than 1. Its dilation gives a tangent vector with a non-zero derivative of the Coulomb energy since the dilation strictly increases all pairwise distances between the points. □

[1] Indices are modulo n, so in the summation we assume $p_{n+1} = p_1$.

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