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Killing Initial Data on spacelike conformal boundaries☆

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ABSTRACT

We analyze Killing Initial Data on Cauchy surfaces in conformally rescaled vacuum space–times satisfying Friedrich's conformal field equations. As an application, we derive the KID equations on a spacelike $\mathscr{I}^-$.

1. Introduction

Symmetries are of utmost importance in physics, and so is the construction of space–times $(\tilde{\mathscr{M}}, \tilde{g})$ satisfying Einstein's field equations in general relativity which possess k-parameter groups of isometries, $1 \leq k \leq 10$ when $\dim \tilde{\mathscr{M}} = 4$, generated by so-called Killing vector fields. Indeed, such space–times can be systematically constructed in terms of an initial value problem when the usual constraint equations, which need to be fulfilled by appropriately prescribed initial data, are supplemented by certain additional equations, the Killing Initial Data (KID) equations.

The KID equations have been derived on spacelike as well as characteristic initial surfaces (cf. [1,2] and references therein). In [3] the same issue was analyzed for characteristic surfaces in conformally rescaled vacuum space–times satisfying Friedrich's conformal field equations. In particular, for vanishing cosmological constant, the KID equations on a light-cone with vertex at past timelike infinity have been derived there. The aim of this work is to carry out the corresponding analysis on spacelike hypersurfaces in conformally rescaled vacuum space–times. As a special case we shall derive the KID equations on $\mathscr{I}^-$ supposing that the cosmological constant is positive so that $\mathscr{I}^-$ is a spacelike hypersurface.

In Section 2 we recall the conformal field equations, discuss their gauge freedom and derive the constraint equations induced on $\mathscr{I}^-$. Well-posedness of the Cauchy problem for the conformal field equations with data on $\mathscr{I}^-$ was shown in [4], we shall provide an alternative proof based on results proved in the Appendix by using a system of wave equations.

The "unphysical Killing equations", introduced in [3] replace, and are in fact equivalent to, the original-space–time Killing equations in the unphysical space–time. Employing results in [3] we derive in Section 3 necessary-and-sufficient conditions on a spacelike hypersurface in a space–time satisfying the conformal field equations which guarantee existence of a vector field fulfilling these equations (cf. Theorem 3.3). Similar to the procedure in [2,3] we first derive an intermediate result, Theorem 3.1, with a couple of additional hypotheses, which then are shown to be automatically satisfied.

In Section 4 we apply Theorem 3.3 to the special case where the spacelike hypersurface is $\mathscr{I}^-$. We shall see that some of the KID equations determine a set of candidate fields on $\mathscr{I}^-$. Whether or not these fields extend to vector fields satisfying the unphysical Killing equations depends on the remaining "reduced KID equations". As for a light-cone with vertex at past

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timelike infinity it turns out that the KID equations adopt at infinity a significantly simpler form as compared to "ordinary" Cauchy surfaces.

Basically, our main result, Theorem 4.1, states the following: An initial data set ($h_{ij} := g_{ij}|_{\mathscr{I}^-}$, $D_{ij} := d_{titj}|_{\mathscr{I}^-}$) for the conformal field equations, with h a Riemannian 3-metric and D a TT-tensor which corresponds to certain components of the conformally rescaled Weyl tensor, generates a $\lambda > 0$-vacuum space–time with a Killing vector field if and only if there exists a conformal Killing vector field $\mathring{X}$ on $(\mathscr{I}^-, h)$ which satisfies the reduced KID equations

$$\mathscr{L}_{\mathring{X}} D + \frac{1}{3}\mathrm{div}(\mathring{X})\, D = 0. \tag{1.1}$$

2. Setting

2.1. Conformal field equations

In $3+1$ dimensions Friedrich's *metric conformal field equations (MCFE)* (cf. [5])[1]

$$\nabla_\rho d_{\mu\nu\sigma}{}^\rho = 0, \tag{2.1}$$

$$\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho \Theta\, d_{\nu\mu\sigma}{}^\rho, \tag{2.2}$$

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu}, \tag{2.3}$$

$$\nabla_\mu s = -L_{\mu\nu}\nabla^\nu \Theta, \tag{2.4}$$

$$2\Theta s - \nabla_\mu \Theta \nabla^\mu \Theta = \lambda/3, \tag{2.5}$$

$$R_{\mu\nu\sigma}{}^\kappa[g] = \Theta d_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu}L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}) \tag{2.6}$$

form a closed system of equations for the unknowns $g_{\mu\nu}$, Θ, s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$. The tensor field $L_{\mu\nu}$ denotes the Schouten tensor,

$$L_{\mu\nu} = \frac{1}{2}R_{\mu\nu} - \frac{1}{12}R g_{\mu\nu}, \tag{2.7}$$

while

$$d_{\mu\nu\sigma}{}^\rho = \Theta^{-1} C_{\mu\nu\sigma}{}^\rho \tag{2.8}$$

is a rescaling of the conformal Weyl tensor $C_{\mu\nu\sigma}{}^\rho$. The function s is defined as

$$s = \frac{1}{4}\Box_g \Theta + \frac{1}{24}R\Theta. \tag{2.9}$$

Friedrich has shown that the MCFE are equivalent to Einstein's vacuum field equations with cosmological constant λ in regions where the conformal factor Θ, relating the "unphysical" metric $g = \Theta^2 g_{\mathrm{phys}}$ with the physical metric g_{phys}, is positive. Their advantage lies in the property that they remain regular even where Θ vanishes.

The system (2.1)–(2.6) treats s, $L_{\mu\nu}$ and $d_{\mu\nu\sigma}{}^\rho$ as independent of $g_{\mu\nu}$ and Θ. However, once a solution of the MCFE has been given these fields are related to $g_{\mu\nu}$ and Θ via (2.7)–(2.9). A solution of the MCFE is thus completely determined by the pair $(g_{\mu\nu}, \Theta)$.

2.2. Gauge freedom

2.2.1. Conformal factor

Let $(g_{\mu\nu}, \Theta, s, L_{\mu\nu}, d_{\mu\nu\sigma}{}^\rho)$ be some smooth solution of the MCFE.[2] From $g_{\mu\nu}$ we compute R. Let us then conformally rescale the metric, $g \mapsto \phi^2 g$, for some positive function $\phi > 0$. The Ricci scalars R and R^* of g and $\phi^2 g$, respectively, are related via (set $\Box_g := g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\phi R - \phi^3 R^* = 6\Box_g \phi. \tag{2.10}$$

Now, let us *prescribe* R^* and read (2.10) as an equation for ϕ. When dealing with a Cauchy problem with data on some spacelike hypersurface $\mathscr{H}$ (including $\mathscr{I}^-$ for $\lambda > 0$) we are free to prescribe functions $\phi|_{\mathscr{H}} =: \mathring{\phi} > 0$ and $\partial_0\phi|_{\mathscr{H}} =: \mathring{\psi}$ on $\mathscr{H}$.[3] Throughout $x^0 \equiv t$ denotes a time-coordinate so that ∂_0 is transverse to $\mathscr{H}$. According to standard results there exists a

[1] It is indicated in [3] that things are considerably different in higher dimensions, which is why we restrict attention to 4 dimensions from the outset.

[2] For convenience we restrict attention throughout to the smooth case, though similar results can be obtained assuming finite differentiability.

[3] The positivity-assumption on $\mathring{\phi}$ makes sure that the solution of (2.10) is positive sufficiently close to $\mathscr{H}$ and thereby that the new conformal factor Θ^* is positive as well (in the $\mathscr{I}^-$-case just off the initial surface).