Journal of Geometry and Physics 106 (2016) 70–86

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Dirac operators on quasi-Hamiltonian G-spaces

Yanli Song

University of Toronto, Canada

ARTICLE INFO

Article history:
Received 22 September 2015
Received in revised form 23 January 2016
Accepted 29 January 2016
Available online 12 February 2016

Keywords:
Dirac operators
Geometric quantization
Quasi-Hamiltonian G-space
Loop group

ABSTRACT

We construct twisted spinor bundles as well as twisted pre-quantum bundles on quasi-Hamiltonian G-spaces, using the spin representation of loop group and the Hilbert space of Wess–Zumino–Witten model. We then define a Hilbert space together with a Dirac operator acting on it. The main result of this paper is that we show the Dirac operator has a well-defined index given by positive energy representation of the loop group. This generalizes the geometric quantization of Hamiltonian G-spaces to quasi-Hamiltonian G-spaces.

1. Introduction

Let G be a compact, connected Lie group, and (M, ω) a compact symplectic manifold with a Hamiltonian G-action. By choosing a G-invariant ω-compatible almost complex structure on M, we can define a G-equivariant $\mathbb{Z}_2$-graded spinor bundle $S_M^{\pm}$. If the Hamiltonian G-space M is pre-quantizable and has a G-equivariant pre-quantum line bundle L, we define a $\mathbb{Z}_2$-graded Hilbert space by

$$\mathcal{H}^{\pm} = L^2(M, S_M^{\pm} \otimes L)$$

and a G-equivariant Spin^c-Dirac operator

$$D^{\pm} : \mathcal{H}^{\pm} \to \mathcal{H}^{\mp}.$$

Attributed to Bott, the *quantization* of (M, ω) can be defined as the equivariant index

$$Q(M, \omega) = \mathrm{Ind}(D) = [\ker(D^+)] - [\ker(D^-)] \in R(G).$$

The goal of this paper is to generalize the quantization process to the *quasi-Hamiltonian G-space* introduced by Alekseev–Malkin–Meinrenken [1]. The q-Hamiltonian G-space, arising from infinite-dimensional Hamiltonian loop group space, differs in many respects from Hamiltonian G-space. In particular, the moment map takes values in the group G and the 2-form ω does not have to be closed or non-degenerate. Consequently, the two key ingredients in defining $Q(M, \omega)$: the spinor bundle S_M and pre-quantum line bundle L might not exist in general.

Given a q-Hamiltonian G-space (M, ω), we use the spin representation of loop group to construct twisted spinor bundles S^{spin} on M, and the Hilbert space of Wess–Zumino–Witten model to construct twisted pre-quantum bundles S^{pre}. Both of them are bundles of Hilbert space and play the same roles as the spinor bundle and pre-quantum line bundle for Hamiltonian

E-mail address: songyanl@math.utoronto.ca.

http://dx.doi.org/10.1016/j.geomphys.2016.01.012

G-space. We analogously define a Hilbert space

$$\mathcal{H} := \left[L^2(M, S^{\text{spin}} \otimes S^{\text{pre}})\right]^G.$$

One key in the construction of Dirac operators on $\mathcal{H}$ is the algebraically defined cubic Dirac operator. It was introduced by Kostant for finite-dimensional Lie group, and extensively studied for infinite-dimensional loop group by various people. Our strategy is to construct a Dirac operator as a combination of algebraic cubic Dirac operators and geometric Spin^c-Dirac operators. To be more precise, we choose an open cover of M using the symplectic cross-section theorem for q-Hamiltonian G-space, so that every open subset U has the geometric structure:

$$U \cong G \times_H V,$$

where H is a compact subgroup of G and the slice V is a Hamiltonian H-space. Accordingly, the tangent bundle TU splits equivariantly into "vertical direction" and "horizontal direction". We define a suitable Dirac operator on U so that it acts as the Spin^c-Dirac operator on the vertical part V and the cubic Dirac operator for loop group on the horizontal part G/H. Using partition of unity, we obtain a global Dirac operator $\mathcal{D}$ on $\mathcal{H}$ by patching together Dirac operators on the open sets U. The main result of this paper is that we show the Dirac operator $\mathcal{D}$ has a well-defined index given by positive energy representations of loop group.

2. Loop group and positive energy representation

We first give a brief review on loop groups and their representations. We use [2] as our primary reference.

2.1. Loop group and central extension

Let G be a compact, simple and simply connected Lie group, and fix a "Sobolev level" $s > 1$. We define LG the *loop group* as the Banach Lie group consisting of maps $S^1 \to G$ of Sobolev class $s + \frac{1}{2}$ with the group structure given by pointwise multiplication. The Lie algebra $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ is given by the space Lie algebra $\mathfrak{g}$-valued 0-forms of Sobolev class $s + \frac{1}{2}$ and $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$ the space of $\mathfrak{g}$-valued 1-forms of Sobolev class $s - \frac{1}{2}$. Integration over S^1 gives a natural non-degenerate pairing between $L\mathfrak{g}$ with $L\mathfrak{g}^*$.

Note that $L\mathfrak{g}^*$ can be identified with the affine space of connections on the trivial principle G-bundle over S^1. The loop group LG acts on $L\mathfrak{g}^*$ by gauge transformation

$$g \cdot \xi = \mathrm{Ad}_g(\xi) - dg \cdot g^{-1}, \quad g \in LG, \xi \in L\mathfrak{g}^*, \tag{2.1}$$

where $dg \cdot g^{-1}$ is the pull-back of the right-invariant Maurer–Cartan form on G.

Let $\widehat{LG}$ be the *basic central extension* of LG, defined in [2, Section 4.4]. The coadjoint action of LG on

$$\widehat{L\mathfrak{g}}^* = L\mathfrak{g}^* \oplus \mathbb{R}$$

is given by the formula

$$g \cdot (\xi, k) = (\mathrm{Ad}_g(\xi) - k \cdot g^{-1} dg, k).$$

One can view the action (2.1) as the coadjoint action on the affine hyperplane $L\mathfrak{g}^* \times \{1\} \subset \widehat{L\mathfrak{g}}^*$.

Fixing a maximal torus T, the choice of a set of positive roots $\mathfrak{R}_+$ for G determines a positive Weyl chamber $\mathfrak{t}^*_+$. It is well-known that the orbits of coadjoint G-action on $\mathfrak{g}^*$ are parameterized by points in $\mathfrak{t}^*_+$. The set of coadjoint LG-orbits can be described as follows. Denote by α_0 the highest root and

$$\rho_G = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_+} \alpha.$$

There is a unique ad-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, rescaled so that the highest root of $\mathfrak{g}$ has norm $\sqrt{2}$. The *dual Coxeter number* of G is defined by

$$h^\vee = 1 + \langle \rho_G, \alpha_0 \rangle_{\mathfrak{g}},$$

and the *fundamental Weyl alcove* for G is the simplex

$$\mathfrak{A} = \{\xi \in \mathfrak{t}_+ \big| \langle \alpha_0, \xi \rangle_{\mathfrak{g}} \leq 1\} \subset \mathfrak{t} \subset \mathfrak{g}.$$

Every coadjoint orbit of LG-action on $L\mathfrak{g}^*$ contains a unique point in $\mathfrak{A}$.

For any $\xi \in L\mathfrak{g}^*$, we define the *holonomy map*

$$\mathrm{Hol} : L\mathfrak{g}^* \to G$$