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A Riemannian approach to Randers geodesics

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ABSTRACT

In certain circumstances tools of Riemannian geometry are sufficient to address questions arising in the more general Finslerian context. We show that one such instance presents itself in the characterisation of geodesics in Randers spaces of constant flag curvature. To achieve a simple, Riemannian derivation of this special family of curves, we exploit the connection between Randers spaces and the Zermelo problem of time-optimal navigation in the presence of background fields. The characterisation of geodesics is then proven by generalising an intuitive argument developed recently for the solution of the quantum Zermelo problem.

Investigations of Finsler manifolds usually require tools more involved than those of Riemannian geometry [1]. For instance, whereas the Levi-Civita connection of Riemannian geometry is a linear connection on the tangent bundle of the underlying manifold, one of its generalisations in the Finslerian context, the so-called Chern connection, is a linear connection on a distinguished vector bundle over the projective sphere bundle [2]. Nevertheless, in certain situations Riemannian methods are sufficient to deal with aspects of Finsler geometry and the resulting simplifications, such as the ones reported below, can be substantial. Specifically, what we show is that the main result of [3], namely, the characterisation of the geodesics of a special class of Finsler spaces, can be proven using tools from Riemannian geometry only.

To begin, let us recall that a Finsler manifold $(\mathcal{M}, F)$ is a C^∞ manifold $\mathcal{M}$ together with a positive function $F(x, y)$ on the tangent bundle, called the Finsler function, which is required to be C^∞ and homogeneous of first degree, that is, $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$. Moreover, the Hessian of F^2 with respect to y:

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^i y^j} F^2(x, y)$$

is assumed to be positive-definite outside the zero-section of $T\mathcal{M}$. It can be shown that $F(x, y) = \sqrt{g_{ij}(x, y) y^i y^j}$.

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If F can be expressed in the form

$$F(x,y)=\sqrt{\alpha_{ij}y^iy^j}+\beta_iy^i,$$

where α is a Riemannian metric and β a one-form, then $\mathcal{M}$ is called a Randers space. The Finslerian metric on $\mathcal{M}$ of Randers type thus takes the form

$$g_{ij}(x,y)=\alpha_{ij}+\beta_i\beta_j+\frac{(\alpha_{ij}\beta_k+\alpha_{jk}\beta_i+\alpha_{ki}\beta_j)y^k}{(\alpha_{kl}y^ky^l)^{1/2}}+\frac{(\beta_ky^k)\alpha_{ik}\alpha_{jl}y^ky^l}{(\alpha_{kl}y^ky^l)^{3/2}}.$$

Randers spaces were first introduced in [4] in the context of a unified theory of gravitation and electromagnetism and arise in a wide range of physical applications such as the electron microscope [5], the propagation of sound and light rays in moving media [6,7], and the time-optimal control in the presence of background fields [8]—the last point being of particular relevance for the present discussion.

To explain the connection between Randers spaces and time-optimal control, we start from a Riemannian manifold $\mathcal{M}$ with metric h, together with a vector field W that satisfies $|W|<1$ and plays the role of the background field, or 'wind'. The goal is to solve the Zermelo problem, that is, to navigate from one point on $\mathcal{M}$ to another along the path $q(s)$ in the shortest possible time under the influence of W, assuming a maximum attainable speed of $|\dot q|=1$ if wind were absent. A problem of this kind was first posed and solved by Zermelo for the navigation of ships at sea (modelled as the Euclidean plane) for a general spacetime-dependent field W [9] (see also [10]). The general formulation on Riemannian manifolds under time-independent fields, and the connection to Randers spaces, was identified more recently by Shen [8]. The idea can be illustrated as follows. Supposing for a moment that one were able to travel for finite time in a tangent space $T_p\mathcal{M}$ for a fixed p, it is clear that the set of destinations reachable in one unit of time coincides with the unit circle, shifted by $W(p)$. Correspondingly, the minimum time $F(p,v)$ it takes to reach the tip of a given vector v in $T_p\mathcal{M}$ is given by the ratio $|v|/|\rho_v|$ of Euclidean norms, where ρ_v is the unique vector collinear with v that lies on the shifted unit circle. To put it differently, the vector $v/F(p,v)-W(p)$ has unit length. It follows that

$$F(p,v)=\frac{-h(v,W(p))+\sqrt{h(v,W(p))^2+|v|^2(1-|W(p)|^2)}}{1-|W(p)|^2}.$$

The function F defined in this manner is a Finsler function of Randers type. Specifically,

$$\alpha_{ij}=\frac{h_{ij}}{1-|W|^2}+\frac{W_iW_j}{(1-|W|^2)^2},\qquad \beta_i=-\frac{W_i}{1-|W|^2},$$

where $W_i=h_{ij}W^j$. Conversely, it can be shown that for each Randers space there is a corresponding Zermelo problem [11]. We remark in passing that there is yet another equivalent perspective, whereby with each Randers space is associated a conformally stationary spacetime [12].

The preceding discussion implies that if a curve $q:[a,b]\to\mathcal{M}$ is traversed at maximum speed, then the time it takes to complete the journey is given by the Randers length

$$T=\int_a^b F(q(s),\dot q(s))\,\mathrm{d}s,$$

where we wrote $\dot q(s)$ for the derivative with respect to the curve parameter s. If the curve $q(s)$ has the physical parameterisation, that is, $q(s)$ corresponds to the location reached by the maximum speed trajectory at time $s-a$ after setting off from $q(a)$, then $F(q(s),\dot q(s))=1$ and $T=b-a$. In other words, curves in the physical parameterisation have unit Randers speed and the passage of time is measured by Randers length. As a consequence, Randers geodesics in the physical parameterisation correspond to solutions of the Zermelo problem. To make this statement more precise, recall that Randers geodesics are curves that locally minimise Randers length. That is, $q:[a,b]\to\mathcal{M}$ is a Randers geodesic if and only if for any $c\in[a,b]$ there exists an interval $I=[c-\varepsilon,c+\varepsilon]$ such that $q|_I$ minimises Randers length among all curves defined on I with the same endpoints. Hence, if endowed with the physical parameterisation, Randers geodesics are the same as curves that locally minimise travel time. Using this equivalence, we can reformulate Theorem 2 of [3] in the following equivalent manner. We write $\mathcal{L}$ for Lie derivative.

Theorem 1. *Assume that the wind vector field W in the Zermelo problem above is an infinitesimal homothety, that is, $\mathcal{L}_Wh=\sigma h$ for a constant σ. Then, if $q:(-\varepsilon,\varepsilon)\to\mathcal{M}$ is a locally time-minimising curve, $p(t)=\varphi_t(t,q(t))$ is a Riemannian geodesic of $(\mathcal{M},h)$, where φ is the flow of $-W$. Conversely, if $p:(-\varepsilon,\varepsilon)\to\mathcal{M}$ is a Riemannian geodesic, $q(t)=\varphi^{-1}(t,p(t))$ is a locally time-minimising curve, where φ^{-1} is the flow of W.*

Notice that the existence of the flow maps on neighbourhoods containing $q(t)$ and $p(t)$, respectively, can be ensured by scaling ε if necessary.

The proof of Theorem 2 of [3] (reformulated here as Theorem 1 above) relies on the geodesic equation for Randers spaces, derived for instance in [2, Chapter 11], which is then verified by explicit calculation—but as we saw above, the Randers geodesics on $(\mathcal{M},F)$ correspond precisely to the locally time-minimising curves of the Zermelo problem. To exploit this fact,