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Generalised time functions and finiteness of the Lorentzian distance

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ABSTRACT

We show that finiteness of the Lorentzian distance is equivalent to the existence of generalised time functions with gradient uniformly bounded away from light cones. To derive this result we introduce new techniques to construct and manipulate achronal sets. As a consequence of these techniques we obtain a functional description of the Lorentzian distance extending the work of Franco (2010) and Moretti (2003).

1. Introduction

This paper originated from asking whether Franco and Moretti's formula for the Lorentzian distance function $d : M \times M \to [0, \infty]$ could be extended to stably causal manifolds, [1,2]. Their proofs were valid only in the globally hyperbolic case. The technical difficulties raised by this problem led to a consideration of the delicate interplay between the Lorentzian distance function, causality and time functions.

Ultimately we were led to develop new techniques for the construction of achronal sets, the manipulation of these sets and a new class of generalised time function. These new techniques allow us to prove our two main results.

Finiteness of the Lorentzian distance. *Let (M, g) be a Lorentzian manifold. The Lorentzian distance is finite if and only if there exists a function $f : M \to \mathbb{R}$, strictly monotonically increasing on timelike curves, whose gradient exists almost everywhere and is such that* $\operatorname{ess\,sup} g(\nabla f, \nabla f) \leq -1$.

The Lorentzian distance formula. *Let (M, g) have finite Lorentzian distance. Then for all $p, q \in M$*

$$d(p, q) = \inf \{\max\{f(q) - f(p), 0\} : f : M \to \mathbb{R},\ f \text{ future directed},\ \operatorname{ess\,sup} g(\nabla f, \nabla f) \leq -1\}. \tag{1}$$

We will refer to equality (1) as the distance formula below.

Franco and Moretti had as their initial motivation the extension of Connes' formula for the Riemannian distance to Lorentzian manifolds, [3], and we note that early investigations and counter-examples appear in [4]. The tools of noncommutative geometry have thus far not been seriously extended past the globally hyperbolic setting, and we hope that our results stimulate further work on this topic.

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The paper is organised as follows. Section 2 summarises those ideas from Lorentzian geometry that we require, and sets notation. In addition we review, and mildly extend, the results of Franco and Moretti. We also prove a 'reverse Lipschitz' characterisation of our generalised time functions in Proposition 2.16, which is essential for applications to Connes-type formulae for the distance.

In brief, the idea of our proof is as follows. Let $S \subset M$ be an achronal set in the Lorentzian manifold (M, g). Then if $M = I^+(S) \cup S \cup I^-(S)$, we can try to define a function $f(x) = d(S, x) = \sup_{s\in S} d(s, x)$ when x is in the future of S, and similarly for other cases. The chief difficulty with this definition is the finiteness of f, even when the Lorentzian distance function d only takes finite values. Much of the difficulty is in finding a suitable set $S \subset M$ with which to define f.

Section 3 contains the technical advances, and is divided into three subsections. The first shows that if the Lorentzian distance is finite then it is possible to choose an achronal subset of the manifold that 'bounds' any divergent behaviour of the metric. The second proves that, under mild assumptions on M, and starting from a suitable achronal set, there exists an achronal surface which divides the manifold into the future of the set, the surface itself and the past of the surface. This is a refinement of a construction of Penrose, [5, Proposition 3.15]. The third section shows how, starting from such a 'bounding' achronal set, to construct a new achronal set. This produces a new achronal set S which separates the manifold M into the future of S, S itself and the past of S. The advantage of this new set is that we can define a generalised time function by taking the Lorentzian distance of a point to S, and this function takes finite values.

Finally, Section 4 presents the proofs of our two main results.

The Appendix provides the details on the regularity of our generalised time functions. A similar concept, also called generalised time functions, has appeared previously, [6]. Our generalised time functions have poor regularity, but in the Appendix we prove that they are continuous almost everywhere, and do have all directional derivatives, and so gradient, existing almost everywhere.

2. Background definitions, notation and results

In the following (M, g) will always be a C^∞, time orientable, path-connected, Lorentzian manifold M of dimension $n + 1 \geq 2$ equipped with a Lorentzian metric g with signature $(-1, 1, \ldots, 1)$. We let T denote the vector field defining the time orientation. The non-time orientable case can be studied via Lorentzian covering manifolds, [7, p 181]. Here and below the measure is always the Lebesgue measure arising from $\sqrt{-\det g}$. Throughout the rest of the paper, unless otherwise noted, we shall use the notation as given in [8]. In particular, for any $U \subset M$, $I^\pm(U) = \bigcup_{x\in U} I^\pm(x)$ and that for any $x \in M$, $I^+(x) = \{y \in M : d(x, y) > 0\}$.

A curve γ is a C^0, piecewise C^1, function from an interval $I \subset \mathbb{R}$ into M so that the tangent vector $\gamma' = \gamma_*(\partial_t)$ is almost everywhere (a.e.) non-zero. For $x, y \in M$ we let $\Omega_{x,y}$ denote the set of future-directed causal curves from x to y. Thus $\gamma \in \Omega_{x,y}$ satisfies $g(\gamma', \gamma') \leq 0$ (causal) everywhere it exists and $g(T, \gamma') < 0$ (future-directed).

By a standard abuse of notation, we sometimes treat γ as a set rather than a curve. Thus $x, y \in \gamma$ means $x, y \in \gamma(I)$, $\gamma \subset U$ means $\gamma(I) \subset U$, and so on. Given a causal curve $\gamma : [a, b] \to M$, the length of γ, denoted $L(\gamma)$ is defined by

$$L(\gamma) = \int_a^b \sqrt{-g(\gamma', \gamma')(t)}dt.$$

Definition 2.1 (*[8, Chapter 4]*)**.** Let (M, g) be a Lorentzian manifold. The Lorentzian distance $d : M \times M \to \mathbb{R}$ is given by

$$d(p, q) := \begin{cases} \sup\limits_{\gamma\in\Omega_{p,q}} L(\gamma) & \Omega_{p,q} \neq \varnothing \\ 0 & \Omega_{p,q} = \varnothing. \end{cases}$$

The Lorentzian distance is always lower semi-continuous, [8, Lemma 4.4].

We make use of the reverse triangle inequality for the Lorentzian distance, [8, page 140]: if $x \in M, y \in I^+(x)$ and $z \in I^+(y)$ then $d(x, z) \geq d(x, y) + d(y, z)$. If for all $x, y \in M$, $d(x, y) < \infty$ then we say that the Lorentzian distance is finite, or that M has finite Lorentzian distance.

Definition 2.2. Let $S \subset M$ be a subset of M. We define the functions $d(S, \cdot) : M \to \mathbb{R} \cup \{\infty\}$ and $d(\cdot, S) : M \to \mathbb{R} \cup \{\infty\}$ by $d(S, x) = \sup\{d(s, x) : s \in S\}$ and $d(x, S) = \sup\{d(x, s) : s \in S\}$.

These functions satisfy a version of the reverse triangle inequality.

Lemma 2.3. *Let $x \in M$, $y \in I^+(x)$ and $S \subset M$. Then:*

1. *$x \in I^+(S)$ implies that $d(S, y) \geq d(S, x) + d(x, y)$;*
2. *$y \in I^-(S)$ implies that $d(x, S) \geq d(x, y) + d(y, S)$.*

Proof. In each case, the reverse triangle inequality implies that:

1. $d(z, y) \geq d(z, x) + d(x, y)$ when $z \in S \cap I^-(x)$;
2. $d(x, z) \geq d(x, y) + d(y, z)$ when $z \in S \cap I^+(y)$.

Taking the supremum over these inequalities with respect to z proves the result. □