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Compact surfaces of constant Gaussian curvature in Randers manifolds☆

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ABSTRACT

The flag curvature of a Finsler surface is called the Gaussian curvature in Finsler geometry. In this paper, we characterize the surfaces of constant Gaussian curvature (CGC) in the Randers 3-manifold. Then we give a classification of the orientable closed CGC surfaces in two Randers space forms, which are the non-Euclidean Minkowski–Randers 3-space ($K = 0$) and the Bao–Shen sphere ($K = 1$).

1. Introduction

A Finsler manifold is a differentiable manifold with a family of norms on its tangent spaces which naturally generalizes the Riemannian manifold. In 1926, L. Berwald extended Riemann's notion of curvature tensor to Finsler geometry and discovered a new non-Riemannian quantity to define the *flag curvature*, which is a natural generalization of the sectional curvature in Riemannian geometry [1]. We call the flag curvature of a 2-dimensional Finsler manifold (i.e., Finsler surface) the *Gaussian curvature*, because it reduces to the classical Gaussian curvature when the Finsler manifold is Riemannian.

A Randers metric is a natural and important Finsler metric which is defined as the sum of a Riemannian metric and a 1-form. It was derived from the research on the general relativity and has been widely applied in many areas of natural science, including biology, physics and psychology, etc. Similar to the Riemannian setting, it is an important problem to study the CGC Finsler surfaces in Randers 3-manifolds. However, it is in general difficult to give a classification even for the CGC surfaces in Riemannian 3-manifolds. In this paper we deal with the closed (i.e. compact without boundary) orientable surfaces of constant Gaussian curvature in Randers 3-manifolds, where the method as well as the results is relatively new. The surfaces considered in this paper are assumed to be connected.

In recent years, people are interested in the surfaces with special properties in certain Riemannian 3-manifold with a Killing vector field, for instance the homogeneous space with isometry group of dimension 4 such as the Berger sphere [2,3]. Suppose that $(\tilde{M}, \tilde{h})$ is a Riemannian 3-manifold and $\tilde{W}$ is a Killing vector field with $\|\tilde{W}\|_{\tilde{h}} < 1$. Consider an orientable

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surface M isometrically immersed in $(\tilde{M}, \tilde{h})$ with N a chosen unit normal vector field. It seems important to consider the *angle function* on M

$$w := \langle N, \tilde{W}\rangle_{\tilde{h}},$$

which was first introduced in studying the surfaces in homogeneous spaces in literatures. For instance, by using the function w, Torralbo and Urbano proved that the Hopf tori are the only closed *flat* surfaces in the Berger sphere, and some nonexistence results were obtained for CGC surfaces in more general homogeneous 3-spaces with isometry group of dimension 4 [3]. Consider an orientable CGC surface M immersed in a Randers 3-manifold with a Killing vector field $\tilde{W}$ and let $\Omega := \{x \in M | w(x) = 0\} \subset M$. The surface M can be separated by two parts, Ω and $M\backslash\Omega$, where Ω is a closed set in M whose interior Int Ω is open and then must consist of pieces of surfaces if it is not empty, and so is the open set $M\backslash\Omega$. By using the angle function w, we give a description of each part separately.

Theorem 1.1. *Let $f : (M^2, F) \to (\tilde{M}^3, \tilde{F})$ be an orientable closed surface isometrically immersed in a Randers manifold with the navigation data $(\tilde{h}, \tilde{W})$, where $\tilde{h}$ is a Riemannian metric and $\tilde{W}$ is a Killing vector field with $\|\tilde{W}\|_{\tilde{h}} < 1$. Denote $\Omega := \{x \in M | w(x) = 0\} \subset M$. Suppose that (M, F) has constant Gaussian curvature $K = \kappa$.*

(a) *If Int $\Omega \neq \emptyset$, then it must consist of pieces of regular surfaces, and each piece must be of constant Gaussian curvature K_h in $(\tilde{M}^3, \tilde{h})$. In this case $\kappa = K_h$ in Int Ω, where $h = f^*\tilde{h}$.*

(b) *If $M\backslash\Omega \neq \emptyset$, then it must consist of pieces of minimal surfaces in $(\tilde{M}^3, \tilde{h})$ satisfying*

$$\left(1 + \frac{1}{1 - w^2}\|W\|_h^2\right)\langle df(\nabla w), \tilde{W}\rangle_{\tilde{h}} + \frac{1}{2}N(\|\tilde{W}\|_{\tilde{h}}^2) = 0, \tag{1}$$

where $\|W\|_h^2$ is the length of the restricting vector W of $\tilde{W}$ on M with respect to $h := f^*\tilde{h}$, and ∇w is the gradient of w with respect to h.

Theorem 1.1 seems quite useful to study the CGC surfaces in *Randers space forms*, i.e., simply connected Randers manifolds of constant flag curvature, which were classified by using the method of Zermelo's navigation [4]. According to the classification, for a certain Riemannian space form and a given Killing vector field with the length less than one, one is able to get a specific Randers space form. For instance, if $\tilde{W}$ is a nonzero constant vector field in the Euclidean space, the corresponding Randers space is called the *non-Euclidean Minkowski–Randers space*. This Minkowski–Randers space is a flat space in the sense that it has vanishing flag curvature, which plays its role in Finsler geometry as the Euclidean space in Riemannian geometry. A Bernstein type theorem has been found in this space [5].

The famous Liebmann theorem states that the only (connected) closed surface in the Euclidean 3-space with constant Gaussian curvature must be a sphere. However, in the non-Euclidean Minkowski–Randers 3-space, we get the following nonexistence result, which is in sharp contract with the Euclidean case.

Theorem 1.2. *There exists no orientable closed surface of constant Gaussian curvature in the non-Euclidean Minkowski–Randers 3-space.*

Besides the non-Euclidean Minkowski–Randers space, we present an important family of Finsler 3-spheres called the *Bao–Shen spheres* introduced in [6] and suggest the Finslerian geometers to study their immersed surfaces. The Bao–Shen sphere is interesting because it has significant properties:

- It is a positively curved Randers space form, i.e., simply connected Randers 3-manifold of constant flag curvature $K = 1$.
- It has the significant physics meaning. Navigating an object in the round 3-sphere under a current of a scaled Reeb vector field along the Hopf fibers, one can find that the shortest time trajectory is the geodesic of a Bao–Shen metric, whose Riemannian part is the Berger metric.
- It has close relation with the round sphere and the Berger sphere whose immersed surfaces are intensively studied in Riemannian geometry.
- It has the Finslerian particularity that it is a Finsler 3-sphere with constant flag curvature $K = 1$ which is not projectively flat, which contrasts with the fact in Riemannian geometry that the spaces forms are projectively flat.

Perhaps it is interesting to keep in our mind that the round sphere and the Berger sphere in Riemannian geometry, as well as the Bao–Shen sphere in Finsler geometry, are much "regular" and worthy to be studied.

It is well known that R^3 has a trivial fibration, and the non-Euclidean Minkowski–Randers space is obtained by perturbing the Euclidean metric with the nonzero constant vector field tangent to the straight line fibers. In the sense of the fibration of base Riemannian space form, the Bao–Shen sphere (by considering the Hopf fibration) is the positively curved analog of the non-Euclidean Minkowski–Randers space. Let us explain the Bao–Shen sphere in a precise way. Consider the standard unit sphere $S^3 \subset R^4 \cong C^2$ with (z, w) the coordinates of C^2. The Reeb vector field is the Killing vector field of unit length $f_1(z, w) = \xi(z, w) = (iz, iw)$, which together with $f_2(z, w) = (-\bar{w}, \bar{z})$ and $f_3(z, w) = (-i\bar{w}, i\bar{z})$ gives an orthonormal frame (f_1, f_2, f_3) on S^3. Suppose the corresponding coframe is $(\zeta^1, \zeta^2, \zeta^3)$, then $\tilde{h}_{S^3} = \sqrt{(\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2}$ is the standard Riemannian metric on S^3.