Journal of Geometry and Physics 106 (2016) 130–142

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Real hypersurfaces in the complex quadric with commuting and parallel Ricci tensor

Young Jin Suh

Kyungpook National University, College of Natural Sciences, Department of Mathematics, Daegu 702-701, Republic of Korea

ARTICLE INFO

Article history:
Received 31 August 2014
Received in revised form 25 February 2016
Accepted 1 March 2016
Available online 9 March 2016

MSC:
primary 53C40
secondary 53C55

Keywords:
Commuting Ricci tensor
Parallel Ricci tensor
$\mathfrak{A}$-isotropic
$\mathfrak{A}$-principal
Complex conjugation
Complex quadric

ABSTRACT

First we introduce the notion of commuting and parallel Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$. Then, according to the $\mathfrak{A}$-isotropic unit normal N, we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_2SO_m$ with commuting and parallel Ricci tensor.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1–4], and [5]). Those are said to be Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we have the example of complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, which is a complex hypersurface in complex projective space $\mathbb{C}P^m$ (see Berndt and Suh [6], and Smyth [7]). The complex quadric also can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [8]). Accordingly, the complex quadric admits two important geometric structures as a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [9] and Reckziegel [10]).

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar\nabla_X A)Y = q(X)AY$ for any vector fields X and Y on Q^m, where $\bar\nabla$ and q denote a connection and a certain 1-form defined on T_zQ^m, $z \in Q^m$ respectively.

E-mail address: yjsuh@knu.ac.kr.

http://dx.doi.org/10.1016/j.geomphys.2016.03.001
0393-0440/

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exists a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

For the complex projective space $\mathbb{C}P^m$ and the quaternionic projective space $\mathbb{H}P^m$ some characterizations was obtained by Okumura [11], and Pérez and Suh [12] respectively. In particular Okumura [11] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. Here the isometric Reeb flow means that $\mathcal{L}_\xi g = 0$ for the Reeb vector field $\xi = -JN$, where N denotes a unit normal vector field of M in $\mathbb{C}P^m$.

For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ the classification was obtained by Berndt and Suh in [1]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$ in [1]. Moreover, in [5] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. In this paper we investigate this problem for the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$. In view of the previous two results a natural expectation might be that the classification involves at least the totally geodesic $Q^{m-1} \subset Q^m$. But, in the complex quadric Q^m Berndt and Suh [6] have proved the following result:

Theorem 1.1. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

By the Kähler structure J of the complex quadric Q^m, we can transfer any tangent vector fields X on M in Q^m as follows:

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ denotes the tangential component of JX and N a unit normal vector field on M in Q^m.

When the Ricci tensor Ric of M in Q^m commutes with the structure tensor ϕ, that is, $Ric \cdot \phi = \phi \cdot Ric$, M is said to be *Ricci commuting*. When the Ricci tensor Ric of M in Q^m is parallel, that is, $\nabla Ric = 0$, let us say M has a *parallel Ricci tensor*. Then first with the notion of *commuting Ricci tensor* for a hypersurface M in the complex quadric Q^m, we can prove the following

Main Theorem 1. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$, with commuting Ricci tensor. Then the unit normal vector field N of M is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

In the first class where M has an $\mathfrak{A}$-isotropic unit normal N, we have asserted in [6] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} if the shape operator commutes with the structure tensor, that is $S \cdot \phi = \phi \cdot S$. In the second class for an $\mathfrak{A}$-principal unit normal N we have proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m if M is a contact hypersurface, that is, $S\phi + \phi S = k\phi$, $k \neq 0$ constant (see [2]).

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z \in M$. Thus for the case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp =$ Span $[A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic.

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$, which is said to be the Reeb function on M.

Then in this paper we give a complete classification for real hypersurfaces in the complex quadric Q^m with commuting and parallel Ricci tensor as follows:

Main Theorem 2. *There do not exist any Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 4$, with commuting and parallel Ricci tensor.*

Now let us consider an Einstein hypersurface in complex quadric Q^m. Then the Ricci tensor of type $(1, 1)$ on M becomes $Ric = \lambda I$, where λ is constant on M and I denotes the identity tensor on M. Accordingly, the Ricci tensor is parallel and commuting, that is Ric $\cdot\, \phi = \phi \,\cdot$ Ric. Moreover, M has an $\mathfrak{A}$-isotropic unit normal vector field N in Q^m. So we assert a corollary as follows:

Corollary 1.2. *There do not exist any Hopf Einstein real hypersurfaces in the complex quadric Q^m, $m \geq 4$.*