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On the formulae for the colored HOMFLY polynomials

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ABSTRACT

We provide methods to compute the colored HOMFLY polynomials of knots and links with symmetric representations based on the linear skein theory. By using diagrammatic calculations, several formulae for the colored HOMFLY polynomials are obtained. As an application, we calculate some examples for hyperbolic knots and links, and we study a generalization of the volume conjecture by means of numerical calculations. In these examples, we observe that asymptotic behaviors of invariants seem to have relations to the volume conjecture.

1. Introduction

This article is devoted to formulae for the colored HOMFLY polynomials of knots and links and its application to the volume conjecture. In general, for a given knot or link, it is difficult to calculate the colored HOMFLY polynomial of it. Therefore, we provide some formulae for the colored HOMFLY polynomials with symmetric representations based on the linear skein theory. These formulae are useful to compute invariants of the knots and links whose diagram has twisted strands with opposite orientations. It is a generalization of the formula of the Jones polynomial [1]. As an application, we explicitly describe invariants of the 5_1 knot, the 6_1 knot, the Whitehead link and the twist knots. Similar invariants are obtained in [2,3]. Furthermore, we take the limits of these invariants in the context of the volume conjecture by numerical calculations. The volume conjecture is first suggested by Kashaev, and formulated by H. Murakami and J. Murakami using the colored Jones polynomial [4,5].

Conjectures 1 (*Volume Conjecture*)**.** *Let L be a hyperbolic link, and let $J_N(L) = J_N(L; q)$ be the colored Jones polynomial associated with the N dimensional irreducible representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$, and let q be* $\exp \frac{2\pi\sqrt{-1}}{N}$. *Then*

$$2\pi \lim_{N\to\infty} \frac{\log J_N(L)}{N} = \mathrm{vol}(L) + \sqrt{-1}\mathrm{CS}(L),$$

where vol *is the hyperbolic volume of the complement of L in $\mathbb{S}^3$ and* CS *is the Chern–Simons invariant of the complement of L in $\mathbb{S}^3$, which is normalized by* $\mathrm{CS}(L) = -2\pi\, cs(L) \bmod \pi^2$ [6,7].

Now, the volume conjecture has been studied by many mathematicians. There are several extensions [8], and the numerical calculations are discussed in [7]. In the second half of this article, we consider another extension. Namely, since the Jones polynomial is extended to the HOMFLY polynomial, we discuss a generalization of the volume conjecture using the HOMFLY polynomials by numerical calculations. Here, according to the feature about the limit of the colored HOMFLY polynomials of the figure-eight knot [9], we calculate invariants of the 5_2 knot, the 6_1 knot, and the Whitehead link by numerical

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calculations. We observe that the asymptotic behaviors of these invariants are similar to that of the figure-eight knot, and that different behavior happens such that there exist limits which does not converge to the volume of corresponding knots and links.

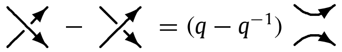
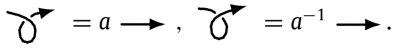
2. Preliminaries

Let a and q be variables in $\mathbb{C}$. We define symbols by

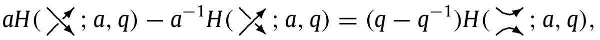
$$[n] = \frac{(q^n - q^{-n})}{(q - q^{-1})}, \qquad [n; a] = \frac{aq^n - a^{-1}q^{-n}}{q - q^{-1}}, \qquad \begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-r+1})}{(1 - q^r)(1 - q^{r-1}) \cdots (1 - q)}.$$

The product is described by descending order with respect to the exponent, and it gives 1 if the product is not defined.

Let F be an oriented compact surface with $2n$ specified points on the boundary. The linear skein of F is the vector space of formal $\mathbb{C}$-linear sums of oriented arcs and link diagrams on F. The arcs consist of n strands and the terminals of the arcs are connected to the $2n$ specified points on ∂F. The linear skein satisfies the following conditions

- regular isotopy,
- $L \cup$ (a trivial closed curve) $= [0; a]L$, and $\varnothing = 1$,
-  (the skein relation),
- 

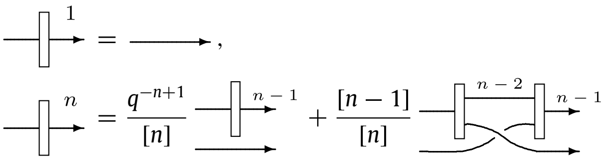
We call a crossing *positive* or *negative* if it is the same crossing appearing in the first or second terms of the skein relation respectively. Let $w(L)$ be the writhe of the oriented arcs and link diagrams L defined by the difference of the numbers of positive and negative crossings of L. When we normalize a link diagram in the linear skein of $\mathbb{S}^2$, with no specified points on the boundary, by $a^{-w(L)}\{(a - a^{-1})/(q - q^{-1})\}^{-1}$, we obtain the HOMFLY polynomial $H(L; a, q)$ [10]. $H(L; a, q)$ is characterized by



$H(L; a, q) = 1, \quad$ where L is a trivial knot.

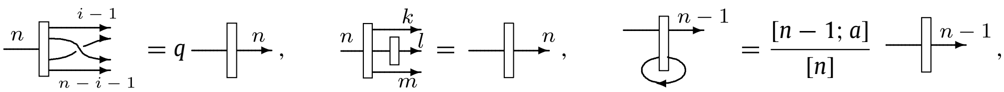
We remark that $H(L; q, q^{-\frac{1}{2}})$ is equal to the Jones polynomial $V_L(q)$ or equivalently $J_2(L; q)$.

An integer n beside the strand indicates n-parallel strands. For an integer $n \geq 1$, an nth q-symmetrizer, denoted by the white rectangle with n, is inductively defined by



$(n \geq 2)$.

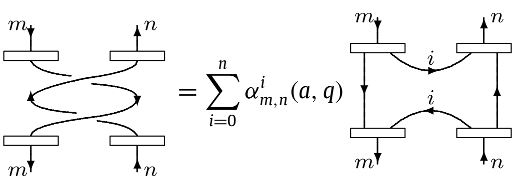
It is well-known that q-symmetrizers have useful properties, which are described by



(1)

where $k + l + m = n$, and the first and second equations hold even if the crossing and the lth q-symmetrizer appear on the left hand side of the nth q-symmetrizer. In what follows, when endpoints appear in a diagram, it means a local diagram.

Lemma 2.1. *For positive integers $m, n (m \geq n)$, the twisted strands can be resolved in the following way.*



where $\alpha^i_{m,n}(a, q)$ denotes

$$\alpha^i_{m,n}(a, q) = (-1)^i a^{-i} (q - q^{-1})^i q^{-i(i-1)} \begin{bmatrix} m \\ 1 \end{bmatrix}_{q^{-2}} \cdots \begin{bmatrix} m - i + 1 \\ 1 \end{bmatrix}_{q^{-2}} \begin{bmatrix} n \\ i \end{bmatrix}_{q^{-2}}.$$