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Recent results on complex Cartan spaces

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ABSTRACT

In this paper, we first provide an updated survey of the geometry of complex Cartan spaces. New characterizations for some particular classes of complex Cartan spaces are pointed out, e.g. Landsberg–Cartan, strongly Berwald–Cartan and others. We introduce the Cartan–Randers spaces which offer examples of Berwald–Cartan and strongly Berwald–Cartan spaces. Then, we investigate the complex geodesic curves of a complex Cartan space, using the image by Legendre transformation ($\mathcal{L}$-duality) of complex geodesic curves of a complex Finsler space. Assuming the weakly Kähler condition for a complex Cartan space, we establish that its complex geodesic curves derive from Hamilton–Jacobi equations. Also, by $\mathcal{L}$-duality, we introduce the corespondent notion of the projectively related complex Finsler metrics, on the complex Cartan spaces. Various descriptions of the projectively related complex Cartan metrics are given. As applications, the projectiveness of a complex Cartan–Randers metric and the locally projectively flat complex Cartan metrics are analyzed.

1. Introduction

The study of geometry of the holomorphic cotangent bundle, endowed with a complex Hamiltonian, has been deepened in some previous works of the second author [1,2]. By analogy with the real case, where remarkable results are known [3–5], the geometry achieved here is called complex Hamilton geometry. The particular context in which the complex Hamiltonian is homogeneous on the fiber, is known as complex Cartan geometry.

The approach of the complex Cartan spaces has been justified by the existence of a pseudo-distance, on the dual holomorphic bundle, highlighted by S. Kobayashi in [6]. Using the equivalence method, J. Faran studied in [7] the complex Cartan spaces, (which he calls Finsler–Hamilton spaces), with constant holomorphic curvature. He also gave some characterizations of the dual Kobayashi metric.

On the other side, as we well know, the Hamiltonian Mechanics can be retrieved via the Lagrangian Mechanics, by so called Legendre transformation. This problem was already extended to the complex case [1], the study of geometric objects on the holomorphic cotangent bundle $T'^{*}M$, via the complex Legendre transformation, defined on the holomorphic tangent bundle $T'M$, being called $\mathcal{L}$-dual process. By $\mathcal{L}$-duality, it is shown that the dual Kobayashi metric is exactly the $\mathcal{L}$-dual of the well-known Kobayashi metric on $T'M$, [1,2].

The $\mathcal{L}$-dual process seems a satisfactory technique for the investigation of the geometry of complex Cartan spaces, using corresponding notions from complex Finsler spaces, for which comprehensive results are known, [8,14–17,10,9,1]. But, a more advertent analysis of previous results, obtained by $\mathcal{L}$-dual process, induced us to come back to some ideas which we reformulated then. More exactly, the $\mathcal{L}$-dual of the vertical natural frame $\frac{\partial}{\partial \eta^k}$ on $T'_u(T'M)$ is identified with the frame that

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is obtained by lifting the subscripts of the vertical natural frame $\frac{\partial}{\partial \zeta_k}$ on $T'_{u^*}(T'^*M)$, only in the purely Hermitian case. This leads us to a lot of difficulties and some new ideas, which we discuss and solve in the present paper.

The paper is organized as follows. After a short survey of complex Cartan spaces in our own notation, (Section 2), we extend some results about classes of complex Cartan spaces obtained in [10]. To the Chern–Cartan complex nonlinear connection, with local coefficients $N_{ji} = -h_{j\bar{k}} \frac{\partial h^{\bar{k}l}}{\partial z^i} \zeta_l$, we associate a complex linear connection of Berwald type $B\Gamma := (N_{ji}, B^i_{jk}, B^{\bar{i}}_{\bar{j}k}, 0, 0)$, which is not of $(1, 0)$-type or metrical compatible. Here, we prove that the conditions: $B\Gamma$ is horizontal metrical compatible and $B\Gamma$ is of $(1, 0)$-type are equivalent (Theorem 3.2) and, we call such a space Landsberg–Cartan. Also, we obtain that any Landsberg–Cartan space with weakly Kähler–Cartan property is a Kähler–Cartan space, (Theorem 3.3). The complex Berwald–Cartan spaces (i.e., the spaces with $B^i_{jk}(z)$) are Landsberg–Cartan. We show that any purely Hermitian complex Cartan space is a complex Berwald–Cartan space. The complex Berwald–Cartan spaces which are weakly Kähler–Cartan are called strongly Berwald–Cartan spaces and they are contained in the class of Kähler–Cartan spaces, (Corollary 3.1). All these results are described in Section 3.

In Section 4, we introduce the Cartan–Randers metrics $\tilde{\mathcal{C}} = \alpha + |\beta|$, where $\alpha = \sqrt{a^{\bar{j}i}(z)\bar{\zeta}_j\zeta_i}$ is a purely Hermitian complex Cartan metric on the complex manifold M and $|\beta|$ is obtained by $\beta = b^i\zeta_i$, $b^i := a^{\bar{j}i}(z)b_{\bar{j}}(z)$, with $b_i(z)$ the local coefficients of a differential $(1, 0)$-form on M. Complex Cartan–Randers metrics are remarkable, they represent the medium in which Hermitian geometry properly interferes with complex Cartan geometry. Theorem 4.2 and Corollary 4.1 report on the necessary and sufficient conditions for a complex Cartan–Randers metric to be a Berwald–Cartan metric or strongly Berwald–Cartan metric. The existence of complex Cartan–Randers spaces with Berwald–Cartan and strongly Berwald–Cartan properties is attested by some explicit examples.

The problem of the complex Cartan spaces obtained as image of the complex Finsler spaces, via complex Legendre transformation is described in Section 5. First of all, we deduce the correct form of the $\mathcal{L}$-dual of vertical natural frame $\frac{\partial}{\partial \eta^k}$, called the nonholonomic vertical frame, (Theorems 5.1, 5.2). Also, we determine the $\mathcal{L}$-dual of Chern–Finsler complex linear connection and by $\mathcal{L}$-duality the weakly Kähler–Finsler property is sent in weakly Kähler–Cartan property. The problem of the complex geodesic curves of a complex Cartan spaces is also investigated by $\mathcal{L}$-dual process. The image $\sigma^*(s)$, by $\mathcal{L}$-duality of the complex geodesics curve $\sigma(s)$ of a complex Finsler spaces is obtained, (Theorem 5.3) and, in the weakly Kähler–Cartan case $\sigma^*(s)$ is a solution of the Hamilton–Jacobi equations. $\sigma^*(s)$ is called the complex geodesic curve of a complex Cartan space and its equations can be rewritten in a more significant form as in Theorem 5.4.

The projectively related complex Cartan spaces are approached by $\mathcal{L}$-duality, too. Two complex Cartan metrics $\tilde{\mathcal{C}}$ and $\mathcal{C}$ on a common underlying manifold M, obtained by $\mathcal{L}$-duality, are called projectively related if any complex geodesic curve, in the sense describe above, of the first is also a complex geodesic curve for the second and vice versa. This means that between the functions $\tilde{N}_k$ and N_k there is a so-called projective change $\tilde{N}_k = N_k + B_k + Q\zeta_k$, where Q is a smooth function on $\widetilde{T'^*M}$ with complex values and $B_k := \tilde{h}_{sk}\tilde{\Theta}^{*s} - h_{sk}\Theta^{*s}$, (Theorem 5.5). Finally, considering a Cartan–Randers metric $\tilde{\mathcal{C}} = \alpha + |\beta|$, we prove that $\tilde{\mathcal{C}}$ can be the image by $\mathcal{L}$-duality of a complex Finsler metric only if it is purely Hermitian. Then, we find the necessary and sufficient conditions under which $\tilde{\mathcal{C}}$ and α are projectively related. Also, the locally projectively flat complex Cartan metrics are pointed out, (Corollary 5.3).

2. Preliminaries

Geometry of real Finsler spaces is already one classic today, ([8,11–13], etc.). During the last years, we remark a significant progress in the study of complex Finsler geometry, ([8,14–17,10,9,1,18,19], etc.). Also, the study of Cartan spaces (real and complex) is enthralling, ([3,4,1], etc.).

Let M be a n-dimensional complex manifold and $z = (z^k)_{k=\overline{1,n}}$ be complex coordinates in a local chart. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l}\eta^l$, $rank(\frac{\partial z'^k}{\partial z^l}) = n$. The dual of $T'M$ is denoted by T'^*M. On the manifold T'^*M, a point u^* is characterized by the coordinates $u^* = (z^k, \zeta_k)_{k=\overline{1,n}}$, and a change of these has the form $z'^k = z'^k(z)$ and$\zeta'_k = \frac{\partial^* z^j}{\partial z'^k}\zeta_j$, $rank(\frac{\partial^* z'^k}{\partial z^l}) = n$. Here and further, we use the notation with star for the partial derivatives with respect to z, on T'^*M, only to distinguish them from those on $T'M$.

Definition 2.1. A *complex Cartan space* is a pair $(M, \mathcal{C})$, where $\mathcal{C} : T'^*M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

(i) $H := \mathcal{C}^2$ is smooth on $\widetilde{T'^*M} := T'^*M \backslash \{0\}$;
(ii) $\mathcal{C}(z, \zeta) \geq 0$, the equality holds if and only if $\zeta = 0$;
(iii) $\mathcal{C}(z, \lambda\zeta) = |\lambda| C(z, \zeta)$ for $\forall \lambda \in \mathbb{C}$;
(iv) the Hermitian matrix $\left(h^{\bar{j}i}(z, \zeta)\right)$ is positive definite, where $h^{\bar{j}i} := \frac{\partial^2 H}{\partial \zeta_i \partial \bar{\zeta}_j}$ is the fundamental metric tensor.

Equivalently, the condition (iv) means that the indicatrix is strongly pseudo-convex.