Journal of Geometry and Physics 106 (2016) 171–183

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Quasi-minimal Lorentz surfaces with pointwise 1-type Gauss map in pseudo-Euclidean 4-space

Velichka Milousheva [a,b,*], Nurettin Cenk Turgay [c]

[a] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. bl. 8, 1113, Sofia, Bulgaria
[b] "L. Karavelov" Civil Engineering Higher School, 175 Suhodolska Str., 1373 Sofia, Bulgaria
[c] Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, 34469 Maslak, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 11 November 2015
Accepted 19 March 2016
Available online 30 March 2016

MSC:
primary 53B30
secondary 53A35
53B25

Keywords:
Pseudo-Euclidean space
Lorentz surface
Quasi-minimal surface
Finite type Gauss map
Parallel mean curvature vector field

ABSTRACT

A Lorentz surface in the four-dimensional pseudo-Euclidean space with neutral metric is called quasi-minimal if its mean curvature vector is lightlike at each point. In the present paper we obtain the complete classification of quasi-minimal Lorentz surfaces with pointwise 1-type Gauss map.

1. Introduction

In the present paper we study Lorentz surfaces in pseudo-Euclidean space $\mathbb{E}^4_2$. A surface is called *minimal* if its mean curvature vector vanishes identically. Minimal surfaces are important in differential geometry as well as in physics. Minimal Lorentz surfaces in $\mathbb{C}^2_1$ have been classified recently by B.-Y. Chen [1]. Several classification results for minimal Lorentz surfaces in indefinite space forms are obtained in [2]. In particular, a complete classification of all minimal Lorentz surfaces in a pseudo-Euclidean space $\mathbb{E}^m_s$ with arbitrary dimension m and arbitrary index s is given.

A natural extension of minimal surfaces are quasi-minimal surfaces. A surface in a pseudo-Riemannian manifold is called *quasi-minimal* (also pseudo-minimal or marginally trapped) if its mean curvature vector is lightlike at each point of the surface [3]. Quasi-minimal surfaces in pseudo-Euclidean space have been very actively studied in the last few years. In [4] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}^4_2$ and gave a complete classification of biharmonic Lorentz surfaces in $\mathbb{E}^4_2$ with lightlike mean curvature vector. Several other families of quasi-minimal surfaces have also been classified. For example, quasi-minimal surfaces with constant Gauss curvature in $\mathbb{E}^4_2$ were classified in [5,6]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [7] and [8]. The classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ is obtained in [9]. In [10] the

* Corresponding author at: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. bl. 8, 1113, Sofia, Bulgaria.
E-mail addresses: vmil@math.bas.bg (V. Milousheva), turgayn@itu.edu.tr (N.C. Turgay).

http://dx.doi.org/10.1016/j.geomphys.2016.03.023
0393-0440/

classification of quasi-minimal rotational surfaces of elliptic, hyperbolic or parabolic type is given. For an up-to-date survey on quasi-minimal surfaces, see also [11].

Another basic class of surfaces in Riemannian and pseudo-Riemannian geometry are the surfaces with parallel mean curvature vector field, since they are critical points of some natural functionals and play important role in differential geometry, the theory of harmonic maps, as well as in physics. Surfaces with parallel mean curvature vector field in Riemannian space forms were classified in the early 1970s by Chen [12] and Yau [13]. Recently, spacelike surfaces with parallel mean curvature vector field in arbitrary indefinite space forms were classified in [14] and [15]. A complete classification of Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{E}^m_s$ is given in [16–18]. A survey on classical and recent results concerning submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [19].

The study of submanifolds of Euclidean or pseudo-Euclidean space via the notion of finite type immersions began in the late 1970's with the papers [20,21] of B.-Y. Chen. An isometric immersion $x : M \to \mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_s$) is said to be of *finite type* [20], if x identified with the position vector field of M in $\mathbb{E}^m$ (or $\mathbb{E}^m_s$) can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, i.e.

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

where x_0 is a constant map, $x_1, x_2, \dots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. More precisely, if $\lambda_1, \lambda_2, \dots, \lambda_k$ are different, then M is said to be of *k-type*. Many results on finite type immersions have been collected in the survey paper [22]. The newest results on submanifolds of finite type are collected in [23].

The notion of finite type immersion is naturally extended to the Gauss map G on M by B.-Y. Chen and P. Piccinni in [24], where they introduced the problem "*To what extent does the type of the Gauss map of a submanifold of* $\mathbb{E}^m$ *determine the submanifold?*". A submanifold M of an Euclidean (or pseudo-Euclidean) space is said to have 1-*type Gauss map* G, if G satisfies $\Delta G = a(G + C)$ for some $a \in \mathbb{R}$ and some constant vector C.

A submanifold M is said to have *pointwise* 1-*type Gauss map* if its Gauss map G satisfies

$$\Delta G = \phi(G + C) \tag{1}$$

for some non-zero smooth function ϕ on M and some constant vector C [25]. A pointwise 1-type Gauss map is called *proper* if the function ϕ is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *first kind* if the vector C is zero. Otherwise, it is said to be of *second kind*.

Classification results on surfaces with pointwise 1-type Gauss map in Minkowski space have been obtained in the last few years. For example, in [26] Y. Kim and D. Yoon studied ruled surfaces with 1-type Gauss map in Minkowski space $\mathbb{E}^m_1$ and gave a complete classification of null scrolls with 1-type Gauss map. The classification of ruled surfaces with pointwise 1-type Gauss map of first kind in Minkowski space $\mathbb{E}^3_1$ is given in [27]. Ruled surfaces with pointwise 1-type Gauss map of second kind in Minkowski 3-space were classified in [28].

The complete classification of flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ is given in [29]. A classification of flat Moore type rotational surfaces in terms of the type of their Gauss map is obtained in [30]. Recently, Arslan and the first author have obtained a classification of meridian surfaces with pointwise 1-type Gauss map [31]. The classification of marginally trapped surfaces with pointwise 1-type Gauss map in Minkowski 4-space is given in [32] and [33].

In the present paper we study quasi-minimal Lorentz surfaces in $\mathbb{E}^4_2$ with pointwise 1-type Gauss map. First we describe the quasi-minimal surfaces with harmonic Gauss map proving that each such surface is a flat surface with parallel mean curvature vector field. Next we give explicitly all flat quasi-minimal surfaces with pointwise 1-type Gauss map (Theorem 3.7). Further, we obtain that a non-flat quasi-minimal surface with flat normal connection has pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field (Theorem 3.8). We give necessary and sufficient conditions for a quasi-minimal surface with non-flat normal connection to have pointwise 1-type Gauss map. In Theorem 3.12 we present the complete classification of quasi-minimal surfaces with non-flat normal connection and pointwise 1-type Gauss map. At the end of the paper we give an explicit example of a quasi-minimal surface with non-flat normal connection and pointwise 1-type Gauss map. This is also an example of a quasi-minimal surface with non-parallel mean curvature vector field.

2. Preliminaries

Let $\mathbb{E}^m_s$ be the pseudo-Euclidean m-space endowed with the canonical pseudo-Euclidean metric of index s given by

$$g_0 = \sum_{i=1}^{m-s} dx_i^2 - \sum_{j=m-s+1}^{m} dx_j^2,$$

where $x_1, x_2, \dots, x_m$ are rectangular coordinates of the points of $\mathbb{E}^m_s$. As usual, we denote by $\langle \, , \, \rangle$ the indefinite inner scalar product with respect to g_0.