Journal of Geometry and Physics 106 (2016) 205–209

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Conformal triality of the Kepler problem

Marco Cariglia

Universidade Federal de Ouro Preto, ICEB, Departamento de Física. Campus Morro do Cruzeiro, Morro do Cruzeiro, 35400-000 - Ouro Preto, MG, Brazil

ARTICLE INFO

Article history:
Received 10 December 2015
Received in revised form 3 March 2016
Accepted 29 March 2016
Available online 8 April 2016

Keywords:
Integrable Hamiltonian systems
Weyl geometry
Null lifts
Projective dynamics

ABSTRACT

We show that the Kepler problem is projectively equivalent to null geodesic motion on the conformal compactification of Minkowski-4 space. This space realises the conformal triality of Minkowski, dS and AdS spaces.

1. Introduction

The Kepler problem can be rightly considered as a ‘golden classic’ in Hamiltonian dynamics for a host of reasons. It is sufficiently realistic to describe within appropriate limits the dynamics of planets in the Solar System, and to some extent classical scattering of pointlike electric charges, while at the same time it is important in the theory of integrable systems, being superintegrable and with a large dynamical symmetry group. It can be taught at a simple level in an undergraduate physics course of mechanics, as well as in a Quantum Mechanics course since integrability holds in the quantum theory too. At the same time a very extensive bibliography and books have been written on the subject and its generalisations, see [1,2] and related works. The literature on the subject being fairly vast, we do not try in this brief letter to cover it in a complete way but rather we try to point out the papers that are most relevant to the line of thought pursued here, and refer the reader to their bibliography to supplement what is missing.

An important moment in the history of the problem comes with the modern, geometrical approach that can be attributed to Fock and then Bargmann [3,4], who analysed the quantum mechanical Kepler problem, and showed that the $SO(4)$ symmetry of negative energy states is explained by mapping the Kepler problem with a fixed energy to that of a free particle on a 3-sphere embedded in 4-dimensional space. The classical description has been discussed and then generalised by Moser and Belbruno. The former showed that the hypersurface of a given negative energy in phase space is homeomorphic to the unit tangent bundle of the sphere S^3 with the north pole excluded, the result being generalisable to general dimension n [5]. Belbruno added the cases of positive energy, in correspondence to the three-hyperboloid $\mathcal{H}^3$, and zero energy which corresponds to 3-dimensional Euclidean space [6].

It is well known that, at the level of the allowed shapes of trajectories of the Kepler problem, ellipses, parabolae and hyperbolae are all related by projective transformations. Thus the question arises if it is possible to find a projective relation between different energy trajectories in the whole phase space. Recent work in the direction of relating geometrically

E-mail address: marco@iceb.ufop.br.

http://dx.doi.org/10.1016/j.geomphys.2016.03.030
0393-0440/

trajectories of different energies has been done by Barrett, Keane and Simmons [7], who showed that the Kepler system is related, for all values of the energy at the same time, to geodesic motion on 3-spaces of constant curvature k via canonical transformations and reparameterisation of trajectories. The relation introduced between the Kepler problem and the geodesic motion on constant curvature manifolds reminds somehow of coupling constant metamorphosis [8]. The two systems described are considered by the authors to be related but different, their Hamiltonians not being the same. Depending on whether k is positive, zero or negative one obtains geometrically the different dynamical symmetry algebras $so(4)$, $iso(3)$ and $so(3,1)$ of the Kepler problem. In [9] 4-dimensional Einstein static spacetimes are studied, which are foliated by the constant curvature 3-spaces mentioned above. These spaces are conformally flat and their conformal group is $SO(4,2)$. Coordinates are found that put the metrics in a manifestly conformally flat form and the Lie algebra of conformal Killing vectors is studied. Null geodesic motion in these space is special since admits the maximum number of conserved charges that are linear in the momenta: they are generated by all the conformal Killing vectors. Keane [10] generalised the constructions and described in detail the zero energy case and time dependent conserved quantities.

The current work is motivated from the desire to give a geometrical interpretation of the full dynamical group $SO(4,2)$ of symmetries of the Kepler problem. An important role in discovering the nature of the above interpretation is played by projective transformations in phase space: as will be seen in the following section, these allow a simple and elegant reformulation of the classical Kepler problem in terms of null geodesics on a conformally flat Weyl space where the dynamical symmetry for all energies is manifest, thus displaying the strength of the technique used and completing the geometrisation of the system. Different conformal classes are projectively related, and in fact one is dealing with null geodesics on the space that realises the conformal triality of Mink, dS and AdS spaces [1,11,12]. In many respects we follow a parallel line to that of Kean, Barrett and Simmons, the main difference being that we use projective geometry to show the dual systems of Kepler and geodesic motion are projectively the same, and that the end result is null dynamics in a 4-dimensional Weyl space that includes at the same time all the values of the curvature: this is the conformal compactification of Minkowski space and as shown it is related to the null lift of Kepler's problem in phase space. In particular, trajectories with different energy sign can be mapped one into the other via a projective transformation. Previous treatments only considered the action of $SO(2,4)$ on standard Kepler trajectories, which does not change the sign of the energy. We also show how to embed trajectories of the original Kepler problem in the space $\mathbb{R}^{2,4}$ which is used to describe the conformal compactification of Minkowski space, realising geometrically the action of $SO(2,4)$ on the projective trajectories.

2. Main result

The Hamiltonian of the Kepler problem is

$$H = \frac{1}{2} p_i p_i - \frac{\alpha}{q}, \tag{2.1}$$

where $\{q^i, p_j\}$ are conjugate variables, $i, j = 1, 2, 3$, $q = \sqrt{q^i q^i}$ and for simplicity in this work we exclude the configuration space point $q = 0$. $\alpha > 0$ is a constant. We consider the following null lift of the Hamiltonian:

$$\mathcal{H} = \frac{1}{2} p_i p_i - \frac{p_y^2}{q} - p_a^2 + p_b^2, \tag{2.2}$$

where we have added new conjugate variables $\{y, p_y\}$, $\{a, p_a\}$, $\{b, p_b\}$, the momenta being conserved. This is a null Hamiltonian that projects to the original one if we impose $\mathcal{H} = 0$, $p_y^2 = \alpha$, and then $p_a^2 = E$, $p_b = 0$ for positive energy solutions of (2.1), $p_a = 0$, $p_b^2 = -E$ for negative energy solutions, and $p_a = 0$, $p_b = 0$ for zero energy solutions. For reasons that will become clear in a few moments, we choose to work in the open phase space region $p_y > 0$. We can exclude $p_y = 0$ since its associated geodesics are related to free motion as it can be seen from (2.2).

In [7] the following dual Hamiltonian is considered

$$G = \frac{1}{4}\left(k + \frac{p_i p_i}{2}\right)^2 q^2, \tag{2.3}$$

which can be canonically transformed into a geodesic Hamiltonian. As the authors note, G above can be formally obtained by solving for α in (2.1), taking its square, setting $-E = k$, and promoting the result to a phase space function. This is quite similar to what happens in coupling constant metamorphosis [8], with the exception of the procedure of taking the square which is novel.

We now relate (2.1) and (2.3) projectively in phase space. As we have shown in a recent companion work [13] any null geodesic Hamiltonian of the form $\frac{1}{2} g^{AB}(q) p_A p_B$, as is $\mathcal{H}$ above, defines a projective conic in tangent space, and any two lower dimensional systems whose null lifts are related one to the other by a projective transformation are dual. In particular, one can rescale the Hamiltonian $\mathcal{H}$ using a phase space factor $\Omega^{-2}(q,p) \neq 0$ and obtain a new Hamiltonian

$$\bar{\mathcal{H}} = \frac{1}{2} \Omega^{-2}(q,p) g^{AB} p_A p_B. \tag{2.4}$$