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Nonassociative geometry in quasi-Hopf representation categories II: Connections and curvature

Gwendolyn E. Barnes, Alexander Schenkel *, Richard J. Szabo

Department of Mathematics, Heriot–Watt University, Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, United Kingdom
Maxwell Institute for Mathematical Sciences, Edinburgh, United Kingdom
The Higgs Centre for Theoretical Physics, Edinburgh, United Kingdom

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ABSTRACT

We continue our systematic development of noncommutative and nonassociative differential geometry internal to the representation category of a quasitriangular quasi-Hopf algebra. We describe derivations, differential operators, differential calculi and connections using universal categorical constructions to capture algebraic properties such as Leibniz rules. Our main result is the construction of morphisms which provide prescriptions for lifting connections to tensor products and to internal homomorphisms. We describe the curvatures of connections within our formalism, and also the formulation of Einstein–Cartan geometry as a putative framework for a nonassociative theory of gravity.

1. Introduction and summary

This paper is the second part in a series of articles whose goal is to systematically develop a formalism for differential geometry on noncommutative and nonassociative spaces. The main physical inspiration behind this work is sparked by the recent observations from closed string theory that certain non-geometric flux compactifications experience a nonassociative deformation of the spacetime geometry [1–6] (see [7–9] for reviews and further references), together with the constructions of [10,11] which show that the corresponding nonassociative algebras and their basic geometric structures can be obtained by cochain twist quantization, and hence are commutative and associative quantities when regarded as objects in a suitable braided monoidal category. See the first paper in this series [12], hereafter referred to as Part I, for further motivation and a more complete list of relevant references.

Earlier categorical approaches to nonassociative geometry along these lines were pursued in [13,14]. In the present paper we develop important notions of differential geometry internal to the representation category ${}^{H}\mathscr{M}$ of a quasitriangular quasi-Hopf algebra H. In particular, we develop the notions of derivations, differential operators, differential calculi and connections by using universal categorical constructions such as categorical limits. In contrast to the approach of [14], our geometric structures are described by internal homomorphisms instead of morphisms in the category ${}^{H}\mathscr{M}$. This leads to a

* Corresponding author.
E-mail addresses: geb31@hw.ac.uk (G.E. Barnes), as880@hw.ac.uk (A. Schenkel), R.J.Szabo@hw.ac.uk (R.J. Szabo).

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much richer framework, because the conditions for being a morphism in ${}^H\mathscr{M}$ (i.e. H-equivariance) are very restrictive and hence the framework in [14] allows for only very special geometric structures. Our internal homomorphism approach is inspired by the formalism of [15] (see [16,17] for overviews), and it clarifies these ideas and constructions in categorical terms.

We begin in Section 2 with a brief review of the categorical framework which was developed in Part I. In contrast to that paper, in the present paper we consider the case where all modules are $\mathbb{Z}$-graded; this allows us later on to regard graded objects such as differential calculi naturally as objects in our categories.

In Section 3 we introduce derivations $\mathrm{der}(A)$ on braided commutative algebras A in ${}^H\mathscr{M}$ by formalizing the Leibniz rule in terms of an equalizer in ${}^H\mathscr{M}$. We analyse structural properties of $\mathrm{der}(A)$ and in particular prove that, in the case where H is triangular, $\mathrm{der}(A)$ together with an internal commutator $[\cdot,\cdot]$ is a Lie algebra in ${}^H\mathscr{M}$. We then introduce differential operators $\mathrm{diff}(V)$ on symmetric A-bimodules V in ${}^H\mathscr{M}$ by again using a suitable equalizer in ${}^H\mathscr{M}$ to capture the relevant algebraic properties. We show that $\mathrm{diff}(V)$ is an algebra in ${}^H\mathscr{M}$ and we also prove that the zeroth order differential operators are the internal endomorphisms $\mathrm{end}_A(V)$ in the category of symmetric A-bimodules ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. Using the product structure on differential operators to formalize nilpotency of a differential, we can then give a definition of a differential calculus in ${}^H\mathscr{M}$.

In Section 4 we develop an appropriate notion of connections $\mathrm{con}(V)$ on objects V in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. The idea is to formalize a generalization of the usual Leibniz rule with respect to a differential calculus in terms of an equalizer in ${}^H\mathscr{M}$. The resulting object $\mathrm{con}(V)$ is analysed in detail and it is shown that the usual affine space of ordinary connections arises as a certain proper subset of $\mathrm{con}(V)$. Our more flexible definition of connections has the advantage that $\mathrm{con}(V)$ also forms an object in ${}^H\mathscr{M}$ in addition to being an affine space. We then develop a lifting prescription for connections to tensor products $V\otimes_A W$ of objects V, W in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. It is important to notice that our notion of tensor product connections differs from the standard one: Although our techniques are only applicable to braided commutative algebras and their bimodules in ${}^H\mathscr{M}$, they are more flexible in the sense that *any* two connections can be lifted to a tensor product connection, not only those which satisfy the very restrictive 'bimodule connection' property proposed in [18–21]. We also develop a lifting prescription for connections to internal homomorphisms $\mathrm{hom}_A(V,W)$ of objects V, W in ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$. These lifts are all important ingredients in (noncommutative and nonassociative) Riemannian geometry for extending e.g. tangent bundle connections to all tensor fields, and they play an instrumental role in physical applications of our formalism to noncommutative and nonassociative gravity theories such as those anticipated to arise in non-geometric string theory. All of these constructions moreover generalize and clarify the corresponding constructions of [15] in categorical terms.

Finally, in Section 5 we assign curvatures to connections and show that they are internal endomorphisms in the category ${}^H{}_A\mathscr{M}_A^{\mathrm{sym}}$, provided that H is triangular. We also obtain a Bianchi tensor, which in classical differential geometry would identically vanish; in general it is not necessarily equal to 0, and hence in this sense it characterizes the noncommutativity and nonassociativity of our geometries. We further observe that the curvature of any tensor product connection is the sum of the two individual curvatures, which means that curvatures behave additively in an appropriate sense. We conclude with a brief outline of how our formalism could be used to describe a noncommutative and nonassociative theory of gravity coupled to Dirac fields; our considerations are based on Einstein–Cartan geometry and its noncommutative generalization which was developed in [22].

2. Categorical preliminaries

Let k be an associative and commutative ring with unit $1\in k$. In contrast to Part I, in this paper we shall work with $\mathbb{Z}$-graded k-modules. This will have the advantage later on that naturally graded objects such as differential calculi can be described as objects in the categories we define below, and also that minus signs will be absorbed into the formalism. The goal of this section is to adapt the material developed in [12] to the graded setting and to thereby also fix our notation for the present paper.

2.1. $\mathbb{Z}$-graded k-modules

The category $\mathscr{M}$ of bounded $\mathbb{Z}$-graded k-modules is defined as follows: The objects in $\mathscr{M}$ are the bounded $\mathbb{Z}$-graded k-modules

$$\underline{V} = \bigoplus_{n\in\mathbb{Z}} \underline{V}_n, \tag{2.1}$$

where the k-modules $\underline{V}_n = 0$ for all but finitely many n. The morphisms in $\mathscr{M}$ are the degree preserving k-linear maps $f:\underline{V}\to\underline{W}$, i.e. $f(\underline{V}_n)\subseteq\underline{W}_n$ for all $n\in\mathbb{Z}$. For any object $\underline{V}$ in $\mathscr{M}$ there is a map

$$|\cdot| : \coprod_{n\in\mathbb{Z}} \underline{V}_n \longrightarrow \mathbb{Z}, \tag{2.2}$$

which assigns to elements $v\in\underline{V}_n$ their degree $|v|=n$. Elements of $\underline{V}_n$ are said to be homogeneous of degree n.