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Null hypersurfaces in generalized Robertson–Walker spacetimes

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ABSTRACT

We study the geometry of null hypersurfaces M in generalized Robertson–Walker spacetimes. First we characterize such null hypersurfaces as graphs of generalized eikonal functions over the fiber and use this characterization to show that such hypersurfaces are parallel if and only if their fibers are also parallel. We further use this technique to construct several examples of null hypersurfaces in both de Sitter and anti de Sitter spaces. Then we characterize all the totally umbilical null hypersurfaces M in a Lorentzian space form (viewed as a quadric in a semi-Euclidean ambient space) as intersections of the space form with a hyperplane. Finally we study the totally umbilical spacelike hypersurfaces of null hypersurfaces in space forms and characterize them as planar sections of M.

1. Introduction

Semi-Riemannian geometry is nowadays a well-established area of research, partly motivated by its applications to General Relativity. Even though semi-Riemannian submanifolds (i.e., those whose induced metric is non-degenerate) have been extensively studied, null submanifolds (with degenerate metric) are less understood, in spite of the fact that numerous features of relevant physical meaning in Relativity find their mathematical grounds in such geometrical objects. That is the case of light trajectories or the smooth parts of event and Cauchy horizons just to name a few.

In spite of their relevance, a systematic study of null submanifolds from a mathematical point of view only flourished from the decade of 1980. Since then, several concepts and results from the semi-Riemannian scenario have been extended to this context, sometimes following different approaches in order to give adequate definitions of the geometrical objects required to study these submanifolds. For example, the Refs. [1–3] provide a broad vision on the subject.

It is also worth noting that the submanifold geometry of null manifolds is yet to be explored in full detail. An example of particular physical interest related to the occurrence of gravitational collapse consists in the study of a spacelike surface S immersed in a null hypersurface M of a four dimensional spacetime $\bar{M}$; see [4–6]. In this setting, S represents the surface of a collapsing massive object (a star for instance) while M is the event horizon associated to the corresponding black hole. Thus, from a geometrical perspective, one of the main problems that arise in this scenario consists in relating the geometrical properties of S and M; in particular, the problem of characterizing the spacelike surfaces subject to suitable geometrical

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restrictions that can be immersed in a null hypersurface of spacetime. A remarkable result in this direction was obtained by Asperti and Dajczer in [7]: For $n \geq 3$, a simply connected n-dimensional Riemannian manifold M may be isometrically immersed in the $(n+1)$-dimensional lightcone if and only if M is conformally flat. As it turns out, even when dealing to the simplest of the null hypersurfaces – namely the light cone in Lorentz–Minkowski space – reveals a rich geometry, as can be seen in the results presented in [8–16]. In particular, in [14], the authors showed that a spacelike hypersurface S of the lightcone in the Minkowski space $\mathbb{R}^{n+2}_1$ is U-totally umbilical with respect to any normal vector field U if and only if S is the intersection of the lightcone with a $(n+1)$-dimensional hyperplane not passing through the origin.

In this paper we will focus in the study of the geometry of null hypersurfaces of a larger class of spacetimes that include all the Lorentzian space forms, namely, the class of generalized Robertson–Walker (GRW) spacetimes. Recall that a GRW spacetime is a Lorentzian warped product $\bar{M} = -I \times_{\varrho} F$, I being a real interval, F a Riemannian manifold and ϱ a differentiable, real, positive function defined on I. If F has constant sectional curvature, then $\bar{M}$ is called a *Robertson–Walker spacetime*. These spacetimes play a key role in Cosmology since they represent the evolution over time of a homogeneous and isotropic universe [4–6]. The class of Robertson–Walker spacetimes includes the de Sitter space $\mathbb{S}^{n+2}_1$ (when F is a sphere) and the anti de Sitter space $\mathbb{H}^{n+2}_1$ (when F is a hyperbolic space), which together with Lorentz–Minkowski space $\mathbb{R}^{n+2}_1$ encompass the class of Lorentzian space forms.

The present work is organized as follows: In Section 2 we establish the notation, definitions and basic structure equations involving null hypersurfaces and their spacelike hypersurfaces. Then in Section 3 we study the geometry of null parallel hypersurfaces in GRW spacetimes. By proving that a submanifold in a GRW spacetime given as the graph of a function $f : F \to \mathbb{R}$ over the fiber is null if and only if it is the graph of a *generalized eikonal* function defined on the fiber we show that such null hypersurface is parallel if and only if its fiber is also parallel; see Theorem 3.8.

In Section 4 we specialize our study to Robertson–Walker spacetimes and use the techniques developed in the previous section to construct concrete examples. Moreover, in Proposition 4.9 we characterize all the totally umbilical null hypersurfaces M in a Lorentzian space form $\bar{M}$ as the intersections of $\bar{M}$ with a hyperplane.

Finally, in Section 5 (Theorem 5.3) we characterize the totally umbilical spacelike hypersurfaces of a totally umbilical null hypersurface M of $\mathbb{S}^{n+2}_1$ and $\mathbb{H}^{n+2}_1$ as the intersections of M with the totally geodesic hypersurfaces of the corresponding Lorentzian space form, thus extending the results presented in [14].

2. Preliminaries

We will follow closely the notation in [17,1,2]. Let $\bar{M}^{n+2}$ be a $(n+2)$-dimensional, semi-Riemannian manifold with metric $\langle\ ,\ \rangle$ and semi-Riemannian connection $\bar{\nabla}$. A submanifold M of $\bar{M}$ is *null* if the restriction of the metric to M is degenerate at each point $p \in M$, which in turn means that for every such p there is a non-zero vector $\xi_p \in T_pM$ such that $\langle \xi_p, X_p\rangle = 0$ for each $X_p \in T_pM$. As usual, if $\dim M = n+1$, we say that M is a *hypersurface* of $\bar{M}$.

Given a null hypersurface $M \subset \bar{M}$, we will consider a *screen distribution* $S(TM)$, that is, a n-dimensional distribution in TM such that the restriction of the metric of M to $S(TM)$ is positive definite. From [1], we know that in a coordinate neighborhood $U \subset M$ there is a vector field N such that

$$\langle \xi, N\rangle = 1, \qquad \langle N, N\rangle = \langle N, X\rangle = 0 \tag{1}$$

for each $X \in \Gamma(S(TM|_U))$, where ξ is a vector field extension of ξ_p to U. We use ξ and N to decompose the tangent bundle $T\bar{M}$ into three vector bundles. First we write $T\bar{M}$ locally as

$$T\bar{M} = TM \oplus \mathrm{span}(N). \tag{2}$$

Additionally, we express TM as

$$TM = S(TM) \oplus_{\mathrm{orth}} \mathrm{span}(\xi), \tag{3}$$

so that

$$T\bar{M} = S(TM) \oplus_{\mathrm{orth}} (\mathrm{span}(\xi) \oplus \mathrm{span}(N)).$$

Let P be the projection of $\Gamma(TM)$ onto $\Gamma(S(TM))$ using the decomposition (3). The local Gauss–Weingarten formulae are

$$\begin{aligned} &\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) = \nabla_X Y + B(X,Y)N, \\ &\bar{\nabla}_X N = -A_N X + \nabla^t_X N = -A_N X + \tau(X)N; \\ &\nabla_X PY = \nabla^*_X PY + h^*(X,PY) = \nabla^*_X PY + C(X,PY)\xi; \\ &\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi = -A^*_\xi X - \tau(X)\xi, \end{aligned} \tag{4}$$

where $X, Y \in \Gamma(TM)$. Here ∇, ∇^t, ∇^* and ∇^{*t} denote the induced connections on TM, $span(N)$, $S(TM)$ and $span(\xi)$, respectively; h and h^* are the second fundamental forms of M and $S(TM)$,

$$B(X,Y) = \langle \bar{\nabla}_X Y, \xi\rangle = \langle A^*_\xi X, Y\rangle,$$

$$C(X,PY) = \langle \nabla_X PY, N\rangle = \langle A_N X, PY\rangle,$$

are the *local second fundamental forms* of M and $S(TM)$, while A_N and A^*_ξ are the *shape operators* on TM and $S(TM)$, respectively. Finally, τ is the 1-form on TM given by $\tau(X) = \langle \bar{\nabla}_X N, \xi\rangle$.