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Solvable groups and a shear construction

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ABSTRACT

The twist construction is a geometric model of T-duality that includes constructions of nilmanifolds from tori. This paper shows how one-dimensional foliations on manifolds may be used in a shear construction, which in algebraic form builds certain solvable Lie groups from Abelian ones. We discuss other examples of geometric structures that may be obtained from the shear construction.

1. Introduction

Recent years have seen a large number of constructions and classifications of geometric structures on nilpotent and solvable Lie groups, for example [1–4]. Many of these structures are motivated by ideas from theoretical physics, and particularly various requirements coming from string and M-theories. An important aspect of such theories are various duality relations. In particular, as Strominger, Yau and Zaslow [5] proposed that T-duality is closely related to the concept of mirror Calabi–Yau manifolds. When fluxes are introduced, the relevant geometries no longer have special holonomy, but are special types of almost Hermitian manifolds in dimension 6 and, for M-theory, G_2-manifolds in dimension 7.

In [6,7] a geometric version of T-duality, called the twist construction, was described that reproduces the known results on nilmanifolds and provides other geometric examples. It was applied in [8] to describe the geometry of the c-map [9] that constructs quaternionic Kähler manifolds in dimension $4n + 4$ from projective special Kähler manifolds in dimension $2n$. The homogeneous models of this construction [10,11] provide all known examples of homogeneous quaternionic Kähler metrics on completely solvable Lie groups [12]. However, the construction of [8] requires modifying the geometry via so-called elementary deformations, before the twist construction is used. This is related to the solvable, rather than nilpotent,

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nature of the homogeneous examples. It is therefore interesting to look for a broader construction that includes the geometry of solvable groups. The purpose of this paper is to propose such a shear construction in the situation where there is a single symmetry, or more generally an appropriate one-dimensional foliation. We will restrict ourselves to a situation which for solvable groups corresponds to having only real eigenvalues. In future work, we will describe how foliations from bundles with flat connections can be used to remove this restriction and consider foliations of higher rank.

In Section 2, we consider the underlying Lie algebraic picture, first recalling the description for nilpotent Lie groups and then extending it to the solvable case. We then provide a general geometric set-up in Section 3 based on a double fibration picture, and discuss carefully what type of bundles may occur. The fibrations will be seen to be given by principal bundles with one-dimensional fibres, but the connection-like one-forms are not necessarily principal. In Section 4, we describe how geometric structures may be transferred through the shear construction, considering which differential forms are naturally related via the horizontal distribution, and providing a formula for the de Rham differentials. Finally Section 5 shows how this shear construction may be applied in examples for different geometric structures.

2. Algebraic models

2.1. The model behind the twist construction is based on the geometry of nilmanifolds. Recall that a Lie algebra $\mathfrak{n}$ is nilpotent if its lower central series $\mathfrak{n}^{(1)} = \mathfrak{n}' = [\mathfrak{n},\mathfrak{n}]$, $\mathfrak{n}^{(k)} = [\mathfrak{n},\mathfrak{n}^{(k-1)}]$, terminates, so $\mathfrak{n}^{(r)} = \{0\}$ for some $r \geqslant 1$. The smallest such number r is called the "step length" of $\mathfrak{n}$. Dually this condition is that there is a minimal filtration $\mathfrak{n}^* = V_0 > V_1 > \cdots > V_{r-1} > \{0\}$, given by $V_i = V_i(\mathfrak{n}) = \operatorname{Ann}(\mathfrak{n}^{(r-i)})$, with $dV_i \subset \Lambda^2 V_{i+1}$ for each i. Our convention is that $d\alpha(X,Y) = -\alpha([X,Y])$; the Jacobi identity is equivalent to $d\circ d = 0$.

Fix an element $\alpha \in V_0 \setminus V_1$ and choose a splitting $\mathfrak{n}^* = \mathbb{R}\alpha \oplus W$ with $V_1 \leqslant W$. Let $F \in \Lambda^2 V_1$ be a two-form with $dF = 0$. Then $dW \leqslant \Lambda^2 V_1 \leqslant \Lambda^2 W$ and we may define a new Lie algebra $\mathfrak{m}$, by taking $\mathfrak{m}^* = \mathbb{R}\beta + W$, retaining the definition of d on W and putting $d\beta = d\alpha + F$. The algebras $\mathfrak{n}$ and $\mathfrak{m}$ are then said to be related by a twist construction.

One valid choice in this construction is $F = -d\alpha$. This gives $d\beta = 0$, and in this case $\mathfrak{m}$ is a Lie algebra direct sum $\mathbb{R}\oplus W^*$, with W^* nilpotent. On the other hand, if $\mathfrak{m}$ is the twist of $\mathfrak{n}$ via F, then we may invert the construction by using the 2-form $-F$. It follows that any nilpotent algebra may be obtained from repeated twists of an Abelian algebra of the same dimension.

Note that dual to the splitting $\mathfrak{n}^* = \mathbb{R}\alpha \oplus W$ we get a unique $X \in \mathfrak{n}$ specified by $W(X) = 0$ and $\alpha(X) = 1$. The condition $V_1 \subset W$ ensures that X is central.

A simple example is provided by the Heisenberg algebra $\mathfrak{h}_3 = (0,0,12)$, where the abbreviated notation means that $\mathfrak{h}^*$ has a basis e_1, e_2, e_3 with corresponding differentials 0, 0, 12, meaning $de_1 = 0$, $de_2 = 0$ and $de_3 = e_1 \wedge e_2 = e_{12}$. In this case, we may take $\alpha = e_3$ and $F = -d\alpha = -e_{12}$. The resulting twist is the Abelian Lie algebra $(0,0,0)$.

2.2. Now consider a solvable algebra $\mathfrak{s}$. This means that the derived series $\mathfrak{s}'$, $(\mathfrak{s}')'$, ... terminates at some finite step. One then has that $\mathfrak{n} = \mathfrak{s}'$ is nilpotent and that $\mathfrak{a} = \mathfrak{s}/\mathfrak{n}$ acts on $\mathfrak{n}$ as an Abelian algebra of endomorphisms. In particular, if $\mathfrak{n}$ has step length r, then the subspace $\mathfrak{n}^{(r-1)}$ is preserved by $\mathfrak{a}$. It follows that the complexification $\mathfrak{n}^{(r-1)} \otimes \mathbb{C}$ contains a one-dimensional invariant subspace $\xi_{\mathbb{C}}$.

For the purposes of this article, let us work in the case when $\xi_{\mathbb{C}}$ may be chosen as the complexification of a real one-dimensional subspace $\xi \leqslant \mathfrak{n}^{(r-1)}$ preserved by $\mathfrak{a}$. For example, this will be the case if $\mathfrak{s}$ is completely solvable.

Fix a basis element X of ξ. Let α be any element of $\mathfrak{s}^*$ with $\alpha(X) = 1$, then for $W = \operatorname{Ann}(\xi)$ we have $\mathfrak{s}^* = \mathbb{R}\alpha \oplus W$.

The choice of ξ implies that there is an element $\eta \in \mathfrak{s}^*$ defined by $[A,X] = \eta(A)X$ for each $A \in \mathfrak{s}$. We now have $d\alpha = \eta\wedge\alpha + F$, with $F \in \Lambda^2 W$ and $\eta \in W$. Furthermore ξ is an ideal of $\mathfrak{s}$ and $dW \subset \Lambda^2 W$. Note that the relation $d^2\alpha = 0$, implies that η and F satisfy

$$dF = \eta\wedge F, \qquad d\eta = 0 \quad \text{and} \quad \eta|_{\xi} = 0. \tag{1}$$

Now suppose $F_0 \in \Lambda^2\mathfrak{s}^*$ is another two-form. We wish to define a new Lie algebra $\mathfrak{r}$ by putting $\mathfrak{r}^* = \mathbb{R}\beta \oplus W$ with $d\beta$ to be essentially $d\alpha + F_0$. More precisely, we may write $F_0 = \eta'\wedge\alpha + F'$, with $\eta', F' \in \Lambda^* W$. Then define

$$d\beta = \tilde{\eta}\wedge\beta + \tilde{F}, \tag{2}$$

with $\tilde{\eta} = \eta + \eta'$ and $\tilde{F} = F + F'$. For this to define a Lie algebra, we need $d^2\beta = 0$, which as above is equivalent to $d\tilde{F} = \tilde{\eta}\wedge\tilde{F}$ and $d\tilde{\eta} = 0$. Translating this back to conditions on F_0, we first note that η' is closed and $\eta' = -X \lrcorner F_0$. Now

$$\begin{aligned} dF_0 &= -\eta'\wedge d\alpha + dF' \\ &= -\eta'\wedge\eta\wedge\alpha - \eta'\wedge F + (\eta+\eta')\wedge(F+F') - \eta\wedge F \\ &= \eta\wedge\eta'\wedge\alpha + (\eta+\eta')\wedge F' = (\eta - X\lrcorner F_0)\wedge F_0. \end{aligned}$$

In other words, we get a Lie algebra if and only if

$$dF_0 = \eta_0\wedge F_0, \qquad d\eta_0 = 0, \qquad \eta_0|_{\xi} = 0 \tag{3}$$

for $\eta_0 = \eta - X\lrcorner F_0$. This is one model of a pair of Lie algebras related by a shear.