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Metrics with prescribed Ricci curvature on homogeneous spaces

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ABSTRACT

Let G be a compact connected Lie group and H a closed subgroup of G. Suppose the homogeneous space G/H is effective and has dimension 3 or higher. Consider a G-invariant, symmetric, positive-semidefinite, nonzero $(0, 2)$-tensor field T on G/H. Assume that H is a maximal connected Lie subgroup of G. We prove the existence of a G-invariant Riemannian metric g and a positive number c such that the Ricci curvature of g coincides with cT on G/H. Afterwards, we examine what happens when the maximality hypothesis fails to hold.

1. Introduction and statement of the main result

The primary objective of the present paper is to produce a global existence theorem for Riemannian metrics with prescribed Ricci curvature on a broad class of compact homogeneous spaces. After stating and proving this theorem, we examine what happens when its key assumption is violated. Let us briefly discuss the history of the subject and describe our results in greater detail.

Suppose M is a smooth manifold. Finding a Riemannian metric g on M whose Ricci curvature $\mathrm{Ric}(g)$ coincides with a prescribed symmetric $(0, 2)$-tensor field T is a fundamental problem in geometric analysis. DeTurck proved the local existence of g in the paper [1] assuming T was nondegenerate on M; see also [2, Chapter 5] and [3, Section 6.5]. Jointly with Goldschmidt, he obtained an analogous result for T of constant rank in [4]. That result required analyticity and several other conditions on T.

Many mathematicians have investigated the global existence of Riemannian metrics with prescribed Ricci curvature. The papers [5–8] provide a snapshot of the recent progress on this topic. We refer to [2, Chapter 5] and [9, Section 9.2] for surveys of older work and to [10] for a sample of the research done in the Lorentzian setting. Most global existence results proven to date deal with open manifolds. However, there are a number of notable exceptions. For instance, Hamilton offered a series of theorems regarding metrics with prescribed Ricci curvature on spheres in [11]. DeTurck and Delanoë obtained more general versions of one of those theorems in [12] and [13].[2] Note that many of the global existence results referenced above share a common feature. Namely, their proofs rely on various forms of the implicit and inverse function theorems.

As far as impact and applications are concerned, DeTurck's work on the paper [1] led him to the discovery of the DeTurck trick for the Ricci flow. Rubinstein showed in [14] that, under natural hypotheses, a sequence of Kähler metrics $(g_i)_{i=1}^{\infty}$ such

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[2] It is unclear from the literature whether Hamilton and DeTurck were aware of each other's work when preparing the papers [11,12].

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that $\mathrm{Ric}(g_{i+1})$ equals g_i for all $i \in \mathbb{N}$ must converge to a Kähler–Einstein metric. He also established a link between $(g_i)_{i=1}^\infty$ and discretisation of geometric flows. Subsequently, he conjectured (personal communication, 30 April 2013) that similar results held for Riemannian $(g_i)_{i=1}^\infty$, at least in some special situations.

Suppose the manifold M is closed. In this case, instead of trying to prove the existence of a metric g with $\mathrm{Ric}(g)$ equal to T, one should search for a metric g and a positive number c such that

$$\mathrm{Ric}(g) = cT. \tag{1.1}$$

This paradigm was originally proposed by Hamilton in [11] and DeTurck in [12]. To explain it, we consider the problem of finding a metric on the 2-dimensional sphere $\mathbb{S}^2$ with prescribed positive-definite Ricci curvature $T_{\mathbb{S}^2}$. According to the Gauss–Bonnet theorem and [11, Theorem 2.1] (see also [15, Corollary 2.2]), such a metric exists if and only if the volume of $\mathbb{S}^2$ with respect to $T_{\mathbb{S}^2}$ equals 4π. Consequently, it is always possible to find g and c such that (1.1) holds on $\mathbb{S}^2$ with T replaced by $T_{\mathbb{S}^2}$. The value of c is uniquely determined by $T_{\mathbb{S}^2}$. Hamilton suggests in [11, Section 1] that the purpose of c is to compensate for the invariance of the Ricci curvature under scaling of the metric.

Consider a compact connected Lie group G and a closed connected subgroup $H < G$ such that the homogeneous space G/H has dimension $n \geq 3$. Let the manifold M coincide with G/H. For simplicity, we assume G acts effectively on M. The following theorem is the main result of the present paper. We prove it in Section 2.

Theorem 1.1. *Suppose H is a maximal connected Lie subgroup of G. Let T be a symmetric G-invariant $(0, 2)$-tensor field on M. Assume T is positive-semidefinite but not identically zero on M. There exist a G-invariant Riemannian metric g and a number $c > 0$ such that Eq. (1.1) holds true.*

Let us make a few remarks. For a comprehensive discussion of concrete examples of homogeneous spaces satisfying the hypotheses of Theorem 1.1, see [16, pages 185–186]. As far as conditions on T are concerned, the majority of global existence results for metrics with prescribed Ricci curvature, including the results in [11–13], require that the prescribed tensor field be positive- or negative-definite. Sometimes, this requirement is implicit. For instance, it is automatically satisfied if the prescribed tensor field is assumed to be "close" to a non-Ricci-flat Einstein metric. Theorem 1.1, on the other hand, applies when T is positive-semidefinite and nonzero. Our arguments break down when T has mixed signature. Specifically, one cannot take advantage of Lemma 2.4 in this case; see Remark 2.5 for details. Note that, according to Bochner's theorem, there are no G-invariant metrics on M with negative-semidefinite Ricci curvature.

Section 3 explores the case where the isotropy representation of M splits into two inequivalent irreducible summands and the maximality assumption on H is violated. Our main result in this section, Proposition 3.1, provides a necessary and sufficient condition for the existence of a G-invariant metric g and a positive number c satisfying (1.1). When this condition holds, the pair (g, c) is unique up to scaling of g. Proposition 3.1 implies that it is always possible to choose the tensor field T in such a way that cT cannot be the Ricci curvature of any G-invariant metric for any number c. Homogeneous spaces whose isotropy representations split into two irreducible summands were carefully studied by Dickinson and Kerr in the paper [17] and by He in the paper [18]; see also Buzano's work [19].

2. Proof of the main result

The method we use to prove Theorem 1.1 may be thought of as the method of Lagrange multipliers. Our reasoning stands on two pillars. The first one is the interpretation of metrics satisfying (1.1) for some $c \in \mathbb{R}$ as critical points of a functional subject to a constraint. This interpretation, given by Lemma 2.1, differs from the variational principle proposed by Hamilton in [11]. The second pillar is the technique invented by Wang and Ziller in [16] to prove the existence of Einstein metrics on homogeneous spaces obeying the hypotheses of Theorem 1.1. This technique was further developed in [20,21]. To make it work for our purposes will require a new estimate on the scalar curvature of a G-invariant metric on M. This estimate is given by Lemma 2.4.

We begin with a few preparatory remarks. Let $\mathcal{M}$ be the space of G-invariant Riemannian metrics on M. This space carries a natural smooth manifold structure; see, e.g., [22, pages 6318–6319]. The scalar curvature $S(g)$ of a metric $g \in \mathcal{M}$ is constant on M. Therefore, we may interpret $S(g)$ as the result of applying a functional $S : \mathcal{M} \to \mathbb{R}$ to $g \in \mathcal{M}$. Note that S is differentiable on $\mathcal{M}$; see, e.g., [2, Corollary 7.39].

If the dimension of the space of G-invariant symmetric $(0, 2)$-tensor fields on M is equal to 1, then the assertion of Theorem 1.1 is easy to prove. In this case, any metric $g \in \mathcal{M}$ satisfies formula (1.1) for some $c \in \mathbb{R}$. Using Bochner's theorem (see [2, Theorem 1.84]), one concludes that c must be positive. In the remainder of Section 2, we assume the dimension of the space of G-invariant symmetric $(0, 2)$-tensor fields on M is 2 or higher. Let T be such a tensor field. Suppose T is positive-semidefinite but not identically zero. Denote by $\mathrm{tr}_g T$ the trace of T with respect to $g \in \mathcal{M}$. We write $\mathcal{M}_T$ for the space of all $g \in \mathcal{M}$ such that $\mathrm{tr}_g T = 1$. The smooth manifold structure on $\mathcal{M}$ induces a smooth manifold structure on $\mathcal{M}_T$. In fact, $\mathcal{M}_T$ is a hypersurface in $\mathcal{M}$. Denote by $S|_{\mathcal{M}_T}$ the restriction of the functional S to $\mathcal{M}_T$. As we will demonstrate below, $S|_{\mathcal{M}_T}$ attains its largest value when the conditions of Theorem 1.1 are satisfied. Further variational properties of $S|_{\mathcal{M}_T}$ are discussed in Remark 3.2.

Lemma 2.1. *A Riemannian metric $g \in \mathcal{M}_T$ satisfies Eq. (1.1) for some $c \in \mathbb{R}$ if and only if it is a critical point of $S|_{\mathcal{M}_T}$.*