Journal of Geometry and Physics 106 (2016) 327–341

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

An extension of the Kadomtsev–Petviashvili hierarchy and its hamiltonian structures

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ARTICLE INFO

Article history:
Received 2 March 2015
Received in revised form 15 March 2016
Accepted 16 April 2016
Available online 24 April 2016

Keywords:
KP hierarchy
Hamiltonian structure
R-matrix

ABSTRACT

In this note we consider a two-component extension of the Kadomtsev–Petviashvili (KP) hierarchy represented with two types of pseudo-differential operators, and construct its Hamiltonian structures by using the *R*-matrix formalism.

1. Introduction

The Kadomtsev–Petviashvili hierarchy plays a fundamental role in the theory of integrable systems. There are several ways to define the KP hierarchy, and one contracted way is via Lax equations of pseudo-differential operators as follows. Let

$$L_{KP} = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots, \quad \partial = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.1}$$

be a pseudo-differential operator with scalar coefficients u_i depending on the spacial coordinate x. The KP hierarchy is composed by the following evolutionary equations

$$\frac{\partial L_{KP}}{\partial t_k} = [(L_{KP}^k)_+, L_{KP}], \quad k = 1, 2, 3, \ldots. \tag{1.2}$$

Here the subscript "+" means to take the differential part of a pseudo-differential operator. The hierarchy (1.2) is known to possess a series of bi-Hamiltonian structures, which can be constructed, for instance, by the *R*-matrix formalism [1].

The KP hierarchy (1.2) has been generalized to multicomponent versions with scalar pseudo-differential operators replaced by matrix-value ones [2–4]. In such generalizations, the pseudo-differential operators are required to admit certain extra constraints, and it is probably why no Hamiltonian structures underlying have been found. Towards overcoming this difficulty, a step was made by Carlet and Manas [5], who solved the constraints in the 2-component case and parameterized the matrix operators with a set of "free" dependent variables. The study of the 2-component KP hierarchy was also motivated by the development of infinite-dimensional Frobenius manifolds in recent years, especially those associated with the bi-Hamiltonian structures for the Toda lattice and the 2-component BKP hierarchies [6–8].

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http://dx.doi.org/10.1016/j.geomphys.2016.04.008

Inspired by the Toda lattice hierarchy, we now consider an extension of the KP hierarchy from the viewpoint of scalar pseudo-differential operators rather than matrix-value ones. More exactly, given a pair of scalar operators

$$P = D + \sum_{i\geq 1} u_i D^{-i}, \qquad \hat{P} = D^{-1}\hat{u}_{-1} + \sum_{i\geq 0} \hat{u}_i D^i \tag{1.3}$$

with D being a "derivation" on some differential algebra that contains functions u_i and $\hat{u}_i$, we want to define the following commutative flows:

$$\frac{\partial}{\partial t_k}(P, \hat{P}) = \left([(P^k)_+, P], [(P^k)_+, \hat{P}] \right), \tag{1.4}$$

$$\frac{\partial}{\partial \hat{t}_k}(P, \hat{P}) = \left([-(\hat{P}^k)_-, P], [-(\hat{P}^k)_-, \hat{P}] \right) \tag{1.5}$$

for $k = 1, 2, 3, \ldots$. It will be seen that if one takes $D = \partial - \varphi$ with φ being an unknown function of x, then the hierarchy (1.4)–(1.5) is well defined (see Section 2). Such kind of hierarchies appeared in the work [9] of Szablikowski and Blaszak as a dispersive counterpart of the Whitham hierarchy (see, for example, [10]). Their version in fact involves N operators of the form $\hat{P}$ with D replaced by $\partial - \varphi_i$ for distinct functions φ_i $(1 \leq i \leq N)$. However, the convergence property of the operators $\hat{P}^k$, which can contain infinitely many positive powers in D, seems not to have been taken into account before. In this note, we will only consider the case $N = 1$, and illustrate that the operators $\hat{P}^k$ as the so-called pseudo-differential operators of the second type introduced by us [11] in recent years. Such kind of operators converge according to a suitable topology. As to be seen, the extended KP hierarchy (1.4)–(1.5) can be reduced to the 2-component BKP hierarchy [2,11] and the constrained KP hierarchy (see, for example, [12–14]) under suitable constraints.

Observe that the flows (1.4)–(1.5) are defined on a subset of the Lie algebra $\mathcal{g}^- \times \mathcal{g}^+$, with $\mathcal{g}^\mp$ being the algebras of pseudo-differential operators of the first and the second types respectively. It is natural to apply the R-matrix scheme to search for Hamiltonian structures underlying (1.4)–(1.5), such as what we have done for the Toda lattice and the 2-component BKP hierarchies [15] (cf. [16,17]). It will be seen that the simple but useful R-matrix found in [15] is also feasible in the current case, so a series of bi-Hamiltonian structures for the hierarchy (1.4)–(1.5) will be derived. Furthermore, such bi-Hamiltonian structures can be naturally reduced to that for the constrained KP hierarchy. Note that the bi-Hamiltonian structure for the constrained KP hierarchy used to be obtained by Oevel and Strampp [18], Cheng [19] and Dickey [20], respectively, with different methods. In a recent paper [21], the central invariants for the bi-Hamiltonian structure have been calculated, which shows that the constrained KP hierarchy is the so-called topological deformation of its dispersionless limit.

This article is organized as follows. In Section 2 we recall the notions of pseudo-differential operators of the first and the second type, and then check in detail that the extended KP hierarchy (1.4)–(1.5) is well defined. In Section 3, we review briefly the R-matrix method for deriving Hamiltonian structures, and apply this method to the hierarchy (1.4)–(1.5) and its reductions. Finally some remarks will be given in Section 4.

2. An extension of the KP hierarchy

2.1. Preliminary notations

Let $\mathcal{A}$ be a commutative algebra, and $\partial : \mathcal{A} \to \mathcal{A}$ be a derivation. The set of pseudo-differential operators is

$$\mathcal{A}((\partial^{-1})) = \left\{ \sum_{i\leq k} f_i \partial^i \mid f_i \in \mathcal{A}, k \in \mathbb{Z} \right\},$$

in which the product is defined by

$$f\partial^i \cdot g\partial^j = \sum_{r\geq 0} \binom{i}{r} f\, \partial^r(g)\, \partial^{i+j-r}, \quad f, g \in \mathcal{A}. \tag{2.1}$$

Clearly one has the commutator $[\partial, f] = \partial(f)$ for any $f \in \mathcal{A}$.

From now on we assume the algebra $\mathcal{A}$ to be a graded one. Namely, $\mathcal{A} = \prod_{i\geq 0} \mathcal{A}_i$, such that

$$\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}, \qquad \partial(\mathcal{A}_i) \subset \mathcal{A}_{i+1}.$$

Denote $\mathcal{D}^- = \mathcal{A}((\partial^{-1}))$, which is called the algebra of pseudo-differential operators of the first type over $\mathcal{A}$. In comparison, by the algebra of pseudo-differential operators of the second type over $\mathcal{A}$ it means

$$\mathcal{D}^+ = \left\{ \sum_{i\in\mathbb{Z}} \sum_{j\geq \max\{0,k-i\}} a_{i,j}\partial^i \mid a_{i,j} \in \mathcal{A}_j, k \in \mathbb{Z} \right\}, \tag{2.2}$$