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Uniqueness of the momentum map

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ABSTRACT

We give a detailed discussion of existence and uniqueness of the momentum map associated to Poisson Lie actions, which was defined by Lu. We introduce a weaker notion of momentum map, called infinitesimal momentum map, which is defined on one-forms and we analyze its integrability to the Lu's momentum map. Finally, the uniqueness of the Lu's momentum map is studied by describing, explicitly, the tangent space to the space of momentum maps.

1. Introduction

In this paper we focus on the study of properties of the momentum map associated to actions of Poisson Lie groups on Poisson manifolds.

The momentum map for actions of Lie groups (without additional structures) on Poisson manifolds provides a mathematical formalization of the notion of conserved quantity associated to symmetries of a dynamical system. The definition of momentum map in this standard setting only requires a *canonical* action (i.e. an action preserving the Poisson structure on the manifold) and its existence is guaranteed whenever the infinitesimal generators of the Lie algebra action are Hamiltonian vector fields (modulo vanishing of a certain Lie algebra cohomology class), as can be seen e.g. in [1, Chapter 10] and [2, Chapter 4]. A generalization of the momentum map to the case of actions of Poisson Lie groups on Poisson manifolds has been introduced by Lu in [3,4]. The associated reduction theory has been studied by Lu (see e.g. [3]) in the case of actions on symplectic manifolds and for actions on Poisson manifolds in [5]. It is worthwhile to mention that Poisson Lie group actions on Poisson manifolds naturally appear in the study of *R*-matrices (see e.g. [6]); thus, studying the properties of the associated momentum map can be useful for the comprehension of the integrable systems associated to the *R*-matrices.

The detailed construction of Lu's momentum map and its basic properties are recalled in the following section. Basically, given a Poisson Lie group (G, π_G) one introduces its dual (G^*, π_{G^*}) and, under fairly general conditions, G^* carries a Poisson action of G called dressing action (and vice versa). The Lie algebra $\mathfrak{g}$ of G is naturally identified with the space of (left)-invariant one-forms on G^*:

$$\theta : \mathfrak{g} \to \Omega^1(G^*)^{G^*} : \xi \mapsto \theta_\xi . \tag{1.1}$$

Given a Poisson manifold (M, π) with a Poisson action of G, a momentum map is a (smooth) Poisson map

$$J : M \to G^*, \tag{1.2}$$

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such that the infinitesimal action $\mathfrak{g} \to \Gamma(TM) : \xi \mapsto \xi_M$ is induced by J via

$$\xi_M = \pi^\sharp(J^*(\theta_\xi)). \tag{1.3}$$

A standard example of a momentum map is the identity map $G^* \to G^*$, which induces the infinitesimal dressing action.

On one hand, the Poisson structure on G gives its Lie algebra a structure of a Lie bialgebra $(\mathfrak{g}, \delta)$ and hence a structure of Gerstenhaber algebra on $\wedge^\bullet \mathfrak{g}$. On the other hand, the Poisson bracket on M makes $\Omega^1(M)$ into a Lie algebra with bracket $[\cdot,\cdot]_\pi$ which also induces a structure of Gerstenhaber algebra on $\Omega^\bullet(M)$. This allows us to observe that the map α, which associates to an element ξ of $\mathfrak{g}$ a one-form $\alpha_\xi = J^*(\theta_\xi)$, can be generalized to a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [\cdot,\cdot]) \longrightarrow (\Omega^\bullet(M), \mathrm{d}_{dR}, [\cdot,\cdot]_\pi). \tag{1.4}$$

We refer to α as the *infinitesimal momentum map* (cf. Section 3.1 and Proposition 3.2). The existence of the infinitesimal momentum map has been discussed in [7] and its rigidity properties have been recently studied in [8] as a generalization of the results obtained in [9]. A reduction theory associated to the infinitesimal momentum map can be found in [10].

The main subject of this paper is the study of the properties of the infinitesimal momentum map and its relation to the usual momentum map. In particular, we show under which conditions it integrates to the usual momentum map. The fact that α is map of Gerstenhaber algebras reduces to two equations

$$\alpha_{[\xi,\eta]} = [\alpha_\xi, \alpha_\eta]_\pi \tag{1.5}$$

$$\mathrm{d}\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi). \tag{1.6}$$

We observe that Eq. (1.6) is a Maurer–Cartan type equation; in fact, when $M = G^*$ it is precisely the Maurer–Cartan equation for the Lie group G^*. When $\Omega^\bullet$ is formal, it admits explicit solution modulo gauge equivalence and we get the following (cf. Theorem 3.8)

Theorem 1.1. *Suppose that M is a Kähler manifold. The set of gauge equivalence classes of $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying the equation*

$$\mathrm{d}\alpha_\xi = \alpha \wedge \alpha \circ \delta(\xi) \tag{1.7}$$

is in bijective correspondence with the set of the cohomology classes $c \in H^1(M, \mathfrak{g}^*)$[1] *satisfying*

$$[c, c] = 0. \tag{1.8}$$

The conditions under which an infinitesimal momentum map α integrates to the momentum map J (cf. Theorem 3.6 for the details) can be summarized in the following

Theorem 1.2. *Let (M,π) be a Poisson manifold and $\alpha : \mathfrak{g} \to \Omega^1(M)$ an infinitesimal momentum map. Suppose that M and G are simply connected and G is compact. Then $\mathscr{D} = \{\alpha_\xi - \theta_\xi,\ \xi \in \mathfrak{g}\}$ generates an involutive distribution on $M \times G^*$ and a leaf $J_{\mathscr{F}}$ of $\mathscr{D}$ is a graph of a momentum map if*

$$\pi(\alpha_\xi, \alpha_\eta) - \pi_{G^*}(\theta_\xi, \theta_\eta)|_{\mathscr{F}} = 0, \quad \xi, \eta \in \mathfrak{g}. \tag{1.9}$$

In Section 3.2 we study concrete cases of this globalization question and prove the existence and uniqueness/non-uniqueness of a momentum map associated to a given infinitesimal momentum map when the dual Poisson Lie group is abelian and, respectively, the Heisenberg group. For the second case the result is as follows (cf. Theorem 3.10).

Theorem 1.3. *Let G be a Poisson Lie group acting on a Poisson manifold M with an infinitesimal momentum map α and such that G^* is the Heisenberg group. Let ξ, η, ζ denote the basis of $\mathfrak{g}$ dual to the standard basis x, y, z of $\mathfrak{g}^*$, with z central and $[x, y] = z$. Then*

$$\pi(\alpha_\xi, \alpha_\eta) = k \tag{1.10}$$

where k is a constant on M. The form α lifts to a momentum map $J : M \to G^*$ if and only if $k = 0$. When $k = 0$ the set of momentum maps with given α is one dimensional with free transitive action of $\mathbb{R}$.

Finally, in the last section, we study the question of infinitesimal deformations of a given momentum map. The main result is Theorem 4.1, which describes explicitly the space tangent to the space of momentum maps at a given point. The main result can be formulated as a statement that the space of momentum maps has a structure of manifold (in an appropriate $\mathscr{C}^\infty$ topology). Let $H : M \to \mathfrak{g}^*$ be a smooth map. We denote by H_ξ the corresponding smooth map on M given by the pullback of H.

[1] Here c can be interpreted as an element of $\mathrm{Hom}(\mathfrak{g}^*, \mathrm{H}^\bullet_{\mathrm{dR}}(M))$, which is a differential graded Lie algebra.