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The prevalence of tori amongst constant mean curvature planes in $\mathbb{R}^3$

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ABSTRACT

Constant mean curvature (CMC) tori in Euclidean 3-space are described by an algebraic curve, called the spectral curve, together with a line bundle on this curve and a point on S^1, called the Sym point. For a given spectral curve the possible choices of line bundle and Sym point are easily described. The space of spectral curves of tori is totally disconnected. Hence to characterise the "moduli space" of CMC tori one should, for each genus g, determine the closure $\overline{\mathcal{P}^g}$ of spectral curves of CMC tori within the spectral curves of CMC planes having spectral genus g. We identify a real subvariety $\mathcal{R}^g$ and a subset $\mathcal{S}^g \subseteq \mathcal{R}^g$ such that $\mathcal{R}^g_{\max} \subseteq \overline{\mathcal{P}^g} \subseteq \mathcal{S}^g$, where $\mathcal{R}^g_{\max}$ denotes the points of $\mathcal{R}^g$ having maximal dimension. The lowest spectral genus for which tori exist is $g = 2$ and in this case $\mathcal{R}^2 = \mathcal{R}^2_{\max} = \overline{\mathcal{P}^2} = \mathcal{S}^2$. For $g > 2$, we conjecture that $\mathcal{R}^g \supsetneq \mathcal{R}^g_{\max} = \mathcal{S}^g$. We give a number of alternative characterisations of $\mathcal{R}^g_{\max}$ and in particular introduce a new integer invariant of a CMC plane of finite type, called its winding number.

1. Introduction

Amongst constant mean curvature (CMC) immersions of the plane into $\mathbb{R}^3$, those of *finite-type* play a special role. They may be described in purely algebraic–geometric terms; there is [1–3] an explicit bijection between CMC planes of finite type and spectral data, consisting of a Sym point $\lambda_0 \in S^1$, a hyperelliptic curve X with an anti-holomorphic involution ρ, a pair of marked points x_0, x_∞ which are exchanged by ρ and a line bundle on X of degree $g + 1$, quaternionic with respect to ρ. Here g denotes the arithmetic genus of the spectral curve X. To specify an immersion we must first provide the data of the spectral curve and a choice of Sym point and then the corresponding isospectral set of CMC immersions is parameterised by a real slice of the Jacobian $\mathrm{Jac}(X)$ of X.

A fundamental result is that all CMC immersions of genus one surfaces into $\mathbb{R}^3$ are of finite type [1,2]. The double-periodicity of the immersion is encoded in periodicity conditions for the spectral curve X, with the line bundle again being freely chosen from a real subspace of $\mathrm{Jac}(X)$. These periodicity conditions are transcendental and it is not obvious even that tori of arbitrary spectral genus exist. In fact $g = 0, 1$ can be easily excluded but for higher spectral genus, existence was established in [4,5].

The space of spectral curves of CMC tori in Euclidean 3-space is totally disconnected. Hence we cannot directly construct the "moduli space" of CMC tori if we wish this space to have some reasonable structure, such as that of a complex analytic space. Instead the natural goal is to determine the closure of spectral curves of CMC tori within the space $\mathcal{H}^g$ of spectral

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curves of CMC immersions of finite type. We identify a subset $\mathcal{S}^g$ of the space of spectral curves of finite type which contains this closure and give a number of equivalent sufficient conditions for a point of $\mathcal{S}^g$ to lie in the closure. We conjecture that these conditions are indeed satisfied by all points of $\mathcal{S}^g$. Proving this appears to require the development of a stronger deformation theory and we hope to return to this in future work.

In stark contrast to the Euclidean situation, spectral curves of CMC tori in the 3-sphere admit a real one-parameter space of deformations, corresponding to changing the mean curvature of the immersion. In [6], we showed that in the spherical case the closure of the set of spectral curves of CMC tori is in fact the entire set of spectral curves of CMC planes of finite type. The Euclidean case is both more difficult and more interesting.

We begin, in Section 2, by summarising the spectral curve correspondence and introducing the aforementioned subsets $\mathcal{P}^g$, $\mathcal{S}^g$, $\mathcal{R}^g$ of the space $\mathcal{H}^g$ of CMC planes of finite type, which play a central role in this paper. In Section 3 we associate to any CMC plane of finite type a meromorphic function $f : \mathbb{P}^1 \to \mathbb{P}^1$, defined up to Möbius transformations, whose degree and winding number prove to be useful invariants for understanding the moduli space $\mathcal{H}^g$ and the tori within it. Theorem 3.2 shows that for positive spectral genus, the degree of f is strictly larger both than one and than the winding number. Theorem 3.5 uses the winding number to give a decomposition of $\mathcal{H}^g \setminus \mathcal{S}^g$ into finitely many open, nonempty and disjoint subsets. Whitham deformations and the role they play in this paper are explained in Section 4. Section 5 contains the statement and proof of our main result, Theorem 5.8. We conclude in Section 6 by analysing subsets of a Grassmannian analogous to, but simpler than, the spaces $\mathcal{P}^g$, $\mathcal{S}^g$ and $\mathcal{R}^g$.

2. Spectral curves of constant mean curvature tori in $\mathbb{R}^3$

We shall make use of the description of constant mean curvature (CMC) immersions of genus one surfaces in $\mathbb{R}^3$ in terms of spectral curve data [2,1,3]. Each such immersion corresponds to a quintuple $(X, \lambda, \rho, \lambda_0, L)$ where X is an algebraic curve, called the *spectral curve*, with a degree two meromorphic function λ, anti-holomorphic involution ρ, and a line bundle L on this curve, which is quaternionic with respect to $\sigma\rho$, where σ is the hyperelliptic involution induced by λ. The function λ is branched at $x_0 = \lambda^{-1}(0)$ and $x_\infty = \lambda^{-1}(\infty)$ whilst the anti-holomorphic involution ρ covers $\lambda \mapsto \bar{\lambda}^{-1}$ and has all points with $|\lambda| = 1$ as fixed points. The remaining data λ_0 is a point on the unit circle or equivalently a pair of marked points $x_1, x_2 \in X$ such that $\lambda(x_1) = \lambda(x_2) = \lambda_0$. Writing P^d for the set of polynomials of degree at most d, we define $\rho^* : P^d \to P^d$ by

$$\rho^*(p(\lambda)) = \bar{\lambda}^d p(\bar{\lambda}^{-1}),$$

and polynomials satisfying $\overline{\rho^* p} = p$ are said to be real with respect to ρ. The reality condition is equivalent to the requirement that for

$$p(\lambda) = x_1 + p_1\lambda + \cdots + p_d\lambda^d$$

we have $p_{d-i} = \overline{p_i}$. The set of polynomials of degree at most d which are real with respect to ρ is denoted by $P^d_{\mathbb{R}}$.

If $(X, \lambda, \rho, \lambda_0, L)$ is the spectral data of a CMC torus then the quadruple $(X, \lambda, \rho, \lambda_0)$ satisfies periodicity conditions. If these conditions are not satisfied, one still obtains from (X, λ, ρ, L) a constant mean curvature immersion of the plane into $\mathbb{R}^3$. Not all constant mean curvature immersions of the plane correspond to spectral curve data as above; those which do are said to be of *finite type*. To consider all constant mean curvature immersions of the plane one would need to study analytic one-dimensional varieties, it is a key result [2,1] however that all doubly-periodic such immersions correspond to *algebraic* curve data, that is are of finite type. Given a hyperelliptic curve with real structure and $\lambda_0 \in S^1$, the line bundle may be chosen from a real g-dimensional family, where g denotes the arithmetic genus of X and as such we see that constant mean curvature immersions of the plane of finite type come in families whose dimension is given by the spectral genus. We restrict our attention to smooth spectral curves, and we may describe such $X = X_a$ in $\mathbb{C}^2$ by an equation of the form

$$y^2 = \lambda a(\lambda) = (-1)^g \lambda \prod_{j=1}^{g} \frac{\bar{\eta}_j}{|\eta_j|}(\lambda - \eta_j)(\lambda - \bar{\eta}_j^{-1}),$$

where a belongs to the space $\mathcal{H}^g \subset P^{2g}$ of polynomials of degree $2g$ satisfying

- the reality condition $\rho^* a = \bar{a}$,
- $\lambda^{-g} a(\lambda) > 0$ for all $\lambda \in S^1$,
- the highest coefficient of a has absolute value 1 and
- the roots of a are pairwise distinct, forcing X_a to be smooth.

The roots $\eta_1, \ldots, \eta_g$ in the punctured open unit disc $B(0, 1) \setminus \{0\} \subseteq \mathbb{C}$ determine a unique $a \in \mathcal{H}^g$ and hence we view $\mathcal{H}^g$ as an open subspace of $\mathbb{C}^g$, with the induced topology.

The periodicity conditions may then be phrased in terms of a pair of meromorphic differentials on the curve X_a. For each $a \in \mathcal{H}^g$, let $\mathcal{B}_a$ denote the real 2-dimensional space of polynomials b of degree $g + 1$ satisfying $b \in P^{g+1}_{\mathbb{R}}$ and such that the meromorphic differential

$$\Theta_b := \frac{b(\lambda)d\lambda}{\lambda y}$$