Journal of Geometry and Physics 106 (2016) 367–382

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Morita equivalence and spectral triples on noncommutative orbifolds

Antti J. Harju *

Helsinki University, Finland
QMU, London, United Kingdom

ARTICLE INFO

Article history:
Received 16 June 2015
Received in revised form 9 March 2016
Accepted 1 April 2016
Available online 20 April 2016

MSC:
16D90
58B34
57R18

Keywords:
Morita equivalence
Spectral triple
Orbifold

ABSTRACT

Let G be a finite group. Noncommutative geometry of unital G-algebras is studied. A geometric structure is determined by a spectral triple on the crossed product algebra associated with the group action. This structure is to be viewed as a representative of a noncommutative orbifold. Based on a study of classical orbifold groupoids, a Morita equivalence for the crossed product spectral triples is developed. Noncommutative orbifolds are Morita equivalence classes of the crossed product spectral triples. As a special case of this Morita theory one can study freeness of the G-action on the noncommutative level. In the case of a free action, the crossed product formalism reduced to the usual spectral triple formalism on the algebra of G-invariant functions.

0. Introduction

We take up the task to develop a noncommutative geometric model for unital algebras subject to a finite group action. The differential geometric objects relevant to this theory are global action orbifolds. These objects may be naturally represented as Lie groupoids, or more precisely, proper étale Lie groupoids, [1]. It was shown in Ref. [2] that with any effective compact proper étale groupoid with a spin structure, one can associate a Dirac spectral triple on the smooth groupoid convolution algebra. So, from this viewpoint, the generalization of noncommutative geometry to G-algebras is straightforward. A major difference between the manifold theory and the orbifold theory is the implementation of an equivalence relation. Namely, orbifolds (considered as Lie groupoids) should be equipped with the geometric Morita equivalence relation which is essentially weaker relation than the G-equivariant diffeomorphism. The geometric Morita equivalence leaves the orbit space of the groupoid invariant, as well as the isotropy types associated with the action. So, one should view an orbifold as a Morita equivalence class of its representative groupoid since the different Morita classes are merely different geometric models for the same orbit space. A particularly important case is an orbifold that is modelled as a manifold subject to a free action of a finite group. This orbifold is Morita equivalent to the unit groupoid of its orbit space. So, the orbifolds subject to free actions should be viewed as manifold objects in the category of Lie groupoids. This is how the geometric Morita equivalent provides a tool to separate smooth objects from those with singularities.

Suppose that a finite group G acts on the unital complex algebra A. This results in two unital associative algebras of interest: the crossed product algebra $G \ltimes A$ and the invariant subalgebra A^G. The philosophy of this project is to view the

* Correspondence to: Helsinki University, Finland.
E-mail address: harjuaj@gmail.com.

http://dx.doi.org/10.1016/j.geomphys.2016.04.002
0393-0440/

algebra $G \ltimes A$ as an equivariant space whereas a spectral triple on $G \ltimes A$ is viewed as a representative of a noncommutative orbifold. We are also assuming that the G-action on the Hilbert space commutes with the Dirac operator which results in a spectral triple on the invariant subalgebra A^G. This level should be viewed as a noncommutative model for the orbit space under the group action. The spectral triple on A^G captures the metric aspects of this theory which is demonstrated in the appendix of this document. These spectral triples also have an important role in the Morita theory introduced in Section 1.

The main goal of this work is to implement a Morita equivalence for spectral triples on noncommutative orbifolds. We shall follow the standard philosophy of noncommutative geometry and translate the geometric Morita equivalence of orbifolds (viewed as Lie groupoids) into the operator theoretic language of spectral triples. Suppose that $G \ltimes X$ is a representative of a global action orbifold and that there is a Dirac spectral triple on its smooth crossed product algebra $(G \ltimes C^\infty(X), L^2(F_\Sigma), \eth)$. If $K \ltimes Y$ is any other global action orbifold that is Morita equivalent to $G \ltimes X$, and if the Morita equivalence ϕ has been fixed, then there is the induced spectral triple $(K \ltimes C^\infty(Y), L^2(\phi_\# F_\Sigma), \phi_\# \eth)$ which was found in [3]. The induced spectral triple is defined in purely geometric means. It is known that once the geometric Morita equivalence ϕ has been fixed, then ϕ gives rise to a pre-C^*-algebra $K \ltimes C^\infty(Y)$- $G \ltimes C^\infty(X)$ bimodule which describes the Morita equivalence on the algebraic level, [4]. This bimodule has a completion to a Morita bimodule for the C^*-algebra completions. In this manuscript we shall see that the assignment $L^2(F_\Sigma) \mapsto L^2(\phi_\# F_\Sigma)$ is, up to a unitary equivalence, the map that induced an isomorphism of the pre-C^*-algebra representation categories, [5]. It will also be shown that the assignment $\eth \mapsto \phi_\# \eth$ will preserve the K-homology class of the underlying Fredholm module, i.e. in terms of operator KK-theory, the approximate sign of $\phi_\# \eth$ is a connection on $L^2(\phi_\# F_\Sigma)$ for the approximate sign of $\eth$. However, these properties alone are too weak to provide axioms for a realistic model for a Morita equivalence of noncommutative orbifolds because the underlying Fredholm module loses information of the Dirac spectrum. The Dirac spectrum carries information of the metric associated with the riemannian structure. More precisely, in the appendix we shall see that in the case of global action orbifolds, the Dirac spectrum captures the geodesic lengths already on the level of spectral triples on the algebras of smooth invariant functions $C^\infty(X)^G$. This is indeed fully compatible with the intuition: the geodesic length is a G-invariant property, and such properties are Morita invariant. Therefore, we need to introduce additional axiom for the Morita equivalence and require that a Morita equivalence operates as a unitary equivalence on the level of invariant spectral triples. This holds in the case of geometric orbifolds, [3].

Notation. This work is a part of the project consisting of [2,3] and this manuscript. The previous parts are purely geometric studies of spectral triples on proper étale groupoids. Although we do not need this generality in this manuscript, we shall continue to work with the Lie groupoid notation for the sake of coherency. If X is an equivariant manifold subject to an action of a finite group G, then we consider this system as the action Lie groupoid $G \times X \rightrightarrows X$, which will be denoted by $G \ltimes X$. The source and target maps are given by $s(g, x) = x$ and $t(g, x) = g \cdot x$ for all $g \in G$ and $x \in X$. We also use $t^{-1}(x) = (G \ltimes X)^x$ and $s^{-1}(x) = (G \ltimes X)_x$.

In the analysis, we shall identify the smooth crossed product algebras $G \ltimes C^\infty(X)$ with the groupoid convolution algebras $C^\infty(G \ltimes X)$. A smooth left Haar system $\mu = \{\mu^x : x \in X\}$ in $G \ltimes X$ is a collection of measures in $G \times X$ parametrized on X, so that

1. The support of μ^x is $(G \ltimes X)^x$.
2. If $f \in C^\infty(G \ltimes X)$, then $x \mapsto \int f d\mu^x$ is smooth.
3. The measures are left-invariant: $\int f(\sigma\tau) d\mu^{s(\sigma)}(\tau) = \int f(\tau) d\mu^{t(\sigma)}(\tau)$.

Here we are concerned with the action groupoids only. In this case we fix the standard Haar measure λ in the finite group G (the counting measure) and then define a Haar system in $G \ltimes X$ by setting

$$\mu = \{\mu^x = \lambda : x \in X\}.$$

The groupoid convolution $C^\infty(G \ltimes X)$ is the unital algebra equipped with the product

$$\begin{aligned}(f \cdot g)(\sigma) &= \int_{\tau \in (G \ltimes X)_{s(\sigma)}} f(\sigma \cdot \tau^{-1}) g(\tau) d\mu^{s(\sigma)}(\tau) \\ &= \frac{1}{\#G} \sum_{\tau \in (G \ltimes X)_{s(\sigma)}} f(\sigma \cdot \tau^{-1}) g(\tau)\end{aligned}$$

where $\#G$ denotes the number of elements in G. The unit is the function $\mathbf{1}_e$ which is the indicator function of the subset $\{e\} \times X$ in $G \times X$ where e is the unit element of G. We shall drop the measure terms $d\mu^{s(\sigma)}$ in the notation. The algebra $C^\infty(G \ltimes X)$ can be equipped with the universal C^*-norm and the groupoid C^*-algebra $C^\star(G \ltimes X)$ is obtained by taking the completion in this norm [6,7].

Convention. An action Lie groupoid $G \ltimes X$ is called a compact action orbifold if G is a finite group, X is a compact connected smooth manifold and the G action on X is effective. This is a nonstandard notation which is adopted for this study because this is exactly the class of Lie groupoids that are relevant in the study of noncommutative geometry of unital G-algebras.