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The evolution equations for regularized Dirac-geodesics

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ABSTRACT

We study the evolution equations for a regularized version of Dirac-geodesics, which are the one-dimensional analogue of Dirac-harmonic maps. We show that for the regularization being sufficiently large, the evolution equations subconverge to a regularized Dirac-geodesic. Finally, we discuss the relation between the regularized and the original problem.

1. Introduction and results

Harmonic maps between Riemannian manifolds [1] are well studied objects in differential geometry. They form a nice variational problem with a rich structure. In its simplest version, harmonic maps manifest as geodesics. The existence of closed geodesics in a closed Riemannian manifold can be studied by various methods and is well understood at present. A powerful method in this case is the so called heat flow method. Here, one deforms a given curve by a heat-type equation. For geodesics, this method was successfully applied by Ottarsson [2]. The same method can also be applied when the domain manifold is not one-dimensional. In this case Eells and Sampson established their famous existence result for harmonic maps under the assumption that the target manifold has non-positive curvature.

An extension of harmonic maps introduced recently are Dirac-harmonic maps [3]. A Dirac-harmonic map is a pair (ϕ, ψ), (with $\phi\colon M \to N$ being a map and ψ a vector spinor) which is a critical point of an energy functional that is motivated from supersymmetric field theories in physics. In physics, this functional is known as supersymmetric sigma model, for the physical background, see [4]. In contrast to the physical literature the spinor fields we are considering here are real-valued and commuting. The Euler–Lagrange equations for Dirac-harmonic maps couple the equation for harmonic maps to spinor fields. They form a weak elliptic system that couples a first and a second order equation. As limiting cases Dirac-harmonic maps contain both harmonic maps and harmonic spinors.

At present many essential results for Dirac-harmonic maps have been obtained. The regularity theory for Dirac-harmonic maps is fully developed, see [5] and [6], the boundary value problem for Dirac-harmonic maps is treated in [7]. In addition, important concepts like an energy identity have also been established [8]. A classification result for Dirac-harmonic maps between surfaces was obtained in [9].

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However, a general existence result for Dirac-harmonic maps is not available at present. Some explicit solutions of the Euler–Lagrange equations for Dirac-harmonic maps are constructed in [10], see also [11]. A general existence result could be obtained in the situation that the Euler–Lagrange equations for Dirac-harmonic maps decouple [12]. Here, the authors assume the existence of a harmonic map ϕ_0 and use methods from index theory to construct a spinor ψ such that one gets a Dirac-harmonic map (ϕ_0, ψ). An existence result for the boundary value problem for Dirac-harmonic maps was recently obtained in [13].

In this article we want to treat the existence question in the most simple situation. Namely, we assume that $M = S^1$ and N being a closed Riemannian manifold. Our approach uses the heat-flow method, but we cannot apply it directly since the Dirac operator is of first order and thus we would not get a reasonable evolution equation. Hence, we first of all consider a regularization of the energy functional for Dirac-geodesics and consider the heat-flow of the regularized functional. Finally, we study the relation between the regularized and the original problem. The basic idea of our approach is to solve a problem that is better to deal with than Dirac-geodesics and still get an existence result for Dirac-geodesics.

Recently, another variant of the heat-flow for Dirac-harmonic maps and Dirac-geodesics has been introduced in [14] and [15].

An approach similar to the one taken in this paper was performed in [16]. There, an extra term $F(\gamma, \psi)$ in the energy functional is considered and depending on the properties of this term existence results are obtained.

Here, we study the energy functional

$$E_\varepsilon(\gamma, \psi) = \frac{1}{2}\int_{S^1} (|\gamma'|^2 + \langle \psi, \not{D}\psi\rangle + \varepsilon|\not{D}\psi|^2)ds, \tag{1.1}$$

where $\gamma : S^1 \to N$, differentiation with respect to the curve parameter s is abbreviated by $'$. Moreover, ψ is a vector spinor, $\not{D}$ the Dirac operator acting on ψ and ε a positive number. This functional is a regularized version of the energy for Dirac-geodesics, which can be obtained in the limit $\varepsilon \to 0$. The idea we pursue is to obtain an existence result for critical points of $E_\varepsilon(\gamma, \psi)$ and study its relation to the original problem.

The Euler–Lagrange equations of $E_\varepsilon(\gamma, \psi)$ are given by:

$$\tau(\gamma) = \mathcal{R}(\gamma, \psi) + \varepsilon\mathcal{R}_c(\gamma, \psi), \tag{1.2}$$
$$\varepsilon\tilde{\Delta}\psi = \not{D}\psi,$$

where τ denotes the tension field of the curve γ and $\tilde{\Delta}$ the Laplacian acting on vector spinors. Note that $\not{D}^2 = -\tilde{\Delta}$ since our domain is one-dimensional. The curvature terms $\mathcal{R}(\gamma, \psi)$ and $\mathcal{R}_c(\gamma, \psi)$ are given by

$$\mathcal{R}(\gamma, \psi) = \frac{1}{2}R^N(\partial_s \cdot \psi, \psi)\gamma', \tag{1.3}$$

$$\mathcal{R}_c(\gamma, \psi) = R^N(\tilde{\nabla}\psi, \psi)\gamma'. \tag{1.4}$$

Here, R^N denotes the Riemann curvature tensor on N, the $\cdot$ refers to Clifford multiplication and $\tilde{\nabla}$ represents the covariant derivative acting on vector spinors.

In order to obtain an existence result for critical points of $E_\varepsilon(\gamma, \psi)$ we use the L^2-gradient flow of this functional, which is given by the following set of coupled evolution equations

$$\frac{\partial\gamma_t}{\partial t} = \tau(\gamma_t) - \mathcal{R}(\gamma_t, \psi_t) - \varepsilon\mathcal{R}_c(\gamma_t, \psi_t), \tag{1.5}$$

$$\frac{\tilde{\nabla}\psi_t}{\partial t} = \varepsilon\tilde{\Delta}\psi_t - \not{D}\psi_t \tag{1.6}$$

together with the initial data $(\gamma(s, 0), \psi(s, 0)) = (\gamma_0(s), \psi_0(s))$.

Eqs. (1.5) and (1.6) form a parabolic system that behaves nicely from an analytical point of view. Since we take the domain manifold to be S^1 we can use the Bochner technique and the maximum principle to derive energy estimates. Finally, we will prove the following

Theorem 1.1. *Assume that $M = S^1$ with fixed spin structure and N is a compact Riemannian manifold without boundary. Then for any sufficiently regular initial data (γ_0, ψ_0) and any $\varepsilon > 0$ there exists a unique smooth solution of (1.5) and (1.6) for all $t \in [0, \infty)$.*

If $\varepsilon \geq 1$, the evolution equations subconverge at infinity, that is, there exists a sequence $t_k \to \infty$ such that $(\gamma_{t_k}, \psi_{t_k}) \to (\gamma_\infty, \psi_\infty)$ in C^2, where $(\gamma_\infty, \psi_\infty)$ is a regularized Dirac-geodesic homotopic to (γ_0, ψ_0).

In the last section we discuss the relation between the regularized and the original problem. We show that for almost all positive ε there is a one to one correspondence between the critical points of $E(\gamma, \psi)$ and $E_\varepsilon(\gamma, \psi)$.

The results presented in this article are part of the author's Ph.D. thesis [17].

Let us now describe the framework for Dirac-geodesics in more detail. Assume that M is a closed curve, for simplicity take $M = S^1$. In addition, let (N, g_{ij}) be a compact, smooth Riemannian manifold without boundary. We will denote the curve