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The geometry of gravitational lensing magnification

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ABSTRACT

We present a definition of unsigned magnification in gravitational lensing valid on arbitrary convex normal neighborhoods of time oriented Lorentzian manifolds. This definition is a function defined at any two points along a null geodesic that lie in a convex normal neighborhood, and foregoes the usual notions of lens and source planes in gravitational lensing. Rather, it makes essential use of the van Vleck determinant, which we present via the exponential map, and Etherington's definition of luminosity distance for arbitrary spacetimes. We then specialize our definition to spacetimes, like Schwarzschild's, in which the lens is compact and isolated, and show that our magnification function is monotonically increasing along any geodesic contained within a convex normal neighborhood.

1. Introduction

Flux is an important observable in gravitational lensing. Suppose F is the observed flux of a light source, which has been increased due to the gravitational focusing by an intervening massive lensing object, and that the hypothetical flux in the absence of the lens would be F_0. Then the magnification factor due to gravitational lensing is given by

$$\mu = \frac{F}{F_0}, \tag{1}$$

ignoring image orientation. This magnification factor is not regarded as an observable because the intrinsic flux F_0 without the lens is unknown in general. The recent report of the first strongly lensed type Ia supernova [1] proves an exception to this rule because such supernovae are standardizable candles. Moreover, it is clear that many more examples of strongly lensed type Ia supernovae will be found by upcoming surveys, so cases of observable lensing magnification will become more common.

From a theoretical point of view, the magnification factor μ is usually considered in terms of the quasi-Newtonian approximation for gravitational lensing (see, e.g., [2,3]), which treats light rays as piecewise straight lines in Euclidean space. However, the proper arena for gravitational lensing is, of course, the Lorentzian spacetime geometry of General Relativity. Thus, it is desirable to generalize the definition of magnification to a spacetime setting and better understand its geometrical meaning. The aim of this article is to do this by reinterpreting the classic definition of luminosity distance in spacetimes due to Etherington [4] in terms of lensing magnification and the van Vleck determinant [5], which we express here in terms of the exponential map. Indeed, we are not the first to promote the use of the exponential map in gravitational lensing; see, e.g., the viewpoint offered in [6]. For the spacetime view of gravitational lensing in general, see, e.g., [7].

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In our understanding of the van Vleck determinant, we have been greatly aided by the two very comprehensive treatments [8,9]. Indeed, most of Sections 3 and 4, on Synge's world function and the van Vleck determinant, can be found in [8] and [9], with one exception, however, that warrants their inclusion here: namely, our use of the exponential map to compute the van Vleck determinant. For this reason, our notation, and several of our proofs, are noticeably different from most of the existing literature. Our definition of magnification (Definition 3) and subsequent focusing theorem (Theorem 1) are to be found in Section 5.

2. Overview of the exponential map and normal coordinates

Let (M, g) be a connected time oriented four-dimensional Lorentzian manifold, with g having signature $(-, +, +, +)$. Let $\mathscr{C}$ be a *convex normal neighborhood of* $p \in M$, that is, a neighborhood any two points q, q' of which are connected by a *unique* distance-minimizing geodesic $\alpha_{qq'}$ lying entirely in $\mathscr{C}$, though there may well be other geodesics between q and q' that leave and then reenter $\mathscr{C}$; the $\alpha_{qq'}$ are usually referred to as "radial geodesics", and we will adopt this terminology henceforth. Furthermore, one can arrange it so that for each $q \in \mathscr{C}$, $\mathscr{C}$ is contained in the normal neighborhood of q provided by the exponential map at q; that such a $\mathscr{C}$ exists at every point on a Lorentzian manifold is proved, e.g., in [10, pp. 133–136]. Thus normal coordinates (x^i) centered at any point $p \in \mathscr{C}$ cover all of $\mathscr{C}$, and we will in fact describe Synge's "world function" below primarily in terms of such coordinates. Next, "$\alpha_{qq'}$" will always denote the unique radial geodesic in $\mathscr{C}$ from q to q', "$\alpha'_{qq'}(t)$" will denote its tangent vector at the point $\alpha_{qq'}(t)$, and we henceforth adopt the convention of parametrizing our radial geodesics to run for unit affine parameter: $\alpha_{qq'}\colon [0, 1] \longrightarrow \mathscr{C}$, with $\alpha_{qq'}(0) = q$ and $\alpha_{qq'}(1) = q'$. Also, we adopt the convention of writing both a curve, and its coordinate representation in a coordinate chart, using the same symbol. Finally, we the Einstein summation convention will be used throughout, with indices labeled 0, 1, 2, 3.

Because of the essential role played for us by the exponential map, we now briefly review some of its properties. Thus, fix $p \in \mathscr{C}$ and recall that any point $q \in \mathscr{C}$ is given by $q = \gamma_V(1) := \exp_p(V)$ for some unique vector $V \in T_pM$, where γ_V is the unique geodesic starting at p in the direction V, and where $\exp_p$ denotes the exponential map at p. Since $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_pM$ to $\mathscr{C}$, any choice of orthonormal basis $\{E_0|_p, \ldots, E_3|_p\} \subset T_pM$ provides us with "normal" coordinates (x^i) defined with respect to that basis. Indeed, expressing any $X \in T_pM$ as $X = X^iE_i|_p$, the diffeomorphism $E\colon T_pM \longrightarrow \mathbb{R}^n$ sending $X \mapsto (X^0, \ldots, X^3)$ composes with $\exp_p^{-1}$ to give

$$q = \gamma_V(1) \overset{\exp_p^{-1}}{\longmapsto} V = V^iE_i|_p \overset{E}{\mapsto} \underbrace{(V^0, \ldots, V^3) := x(q)}_{(x^i)\text{ coordinates of } q}. \tag{2}$$

Thus $x^i(q) = V^i$, and because geodesics γ_V have the scaling property $\gamma_V(t) = \gamma_{tV}(1)$ whenever either side is defined, the geodesic γ_V in the coordinates (x^i) is given by

$$\gamma_V(t) = (tV^0, \ldots, tV^3). \tag{3}$$

Of course, $\gamma_V\colon [0, 1] \longrightarrow \mathscr{C}$ must be our radial geodesic $\alpha_{pq}\colon [0, 1] \longrightarrow \mathscr{C}$. Furthermore, the normal coordinate basis vectors $\{\partial/\partial x^0, \ldots, \partial/\partial x^3\}$ defined with respect to the (x^i) satisfy, by construction,

$$\left.\frac{\partial}{\partial x^i}\right|_p = E_i|_p,$$

hence $g_{ij}(p) = \text{diag}\,(-1, 1, 1, 1)$ (i.e., they are "normal" at p). Because of (3), it also follows that $\Gamma^i_{jk}(p) = 0$, hence also $\partial_i|_p(g_{jk}) = 0$; consult, e.g., [11, Prop. 33, p. 73].

Now we use normal coordinates at p to define "quasi-normal" coordinates at any other point $q \in \mathscr{C}$, as follows. Let $\{E_0|_p, \ldots, E_3|_p\} \subset T_pM$ denote the orthonormal basis with respect to which the normal coordinates (x^i) at p are defined. Given any other point $q \in \mathscr{C}$, let J_i denote the unique Jacobi field along the radial geodesic $\alpha_{pq}\colon [0, 1] \longrightarrow \mathscr{C}$ satisfying $J_i(0) = 0$ and $J_i'(0) = E_i|_p$, where "J_i'" denotes the covariant derivative of J_i along α_{pq}. Observe that $\{J_0(1), \ldots, J_3(1)\} \subset T_qM$ will be linearly independent provided that q is not conjugate to p along α_{pq}. Indeed this must be so, since $\mathscr{C}$ is contained in the normal coordinate chart centered at p, *no* point in $\mathscr{C}$ is conjugate to p along a radial geodesic through p (this is an important point to which we will return later; consult, e.g., [11, Prop. 10, p. 271]). Thus $\{J_0(1), \ldots, J_3(1)\} \subset T_qM$ is a basis, though it need not be orthonormal. In any case, use it to define "quasi-normal" coordinates $(\bar{x}^i)$ centered at $q \in \mathscr{C}$, via the exponential map $\exp_q$ at q, in the same manner as in (2). Then, by construction,

$$\left.\frac{\partial}{\partial \bar{x}^i}\right|_q = J_i(1)$$

for each $i = 0, \ldots, 3$. In fact, each

$$\left.\frac{\partial}{\partial \bar{x}^i}\right|_q = \left.\frac{\partial}{\partial x^i}\right|_q, \tag{4}$$