Journal of Geometry and Physics 100 (2016) 92–95

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On large families of bundles over algebraic surfaces

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ARTICLE INFO

Article history:
Received 21 July 2015
Received in revised form 21 October 2015
Accepted 31 October 2015
Available online 10 November 2015

MSC:
14D20
14J60

Keywords:
Vector bundle
Algebraic surface
Strong Bogomolov inequality

ABSTRACT

The aim of this note is to construct sequences of vector bundles with unbounded rank and discriminant on an arbitrary algebraic surface. This problem, on principally polarized abelian varieties with cyclic Neron–Severi group generated by the polarization, was considered by Nakashima in connection with the Douglas–Reinbacher–Yau conjecture on the Strong Bogomolov Inequality. In particular we show that on any surface, the Strong Bogomolov Inequality SBI_l is false for all $l > 4$.

1. Introduction

Let S be a smooth complex projective algebraic surface. For a vector bundle E on S with Chern classes c_1 and c_2, the discriminant Δ is defined by the following formula:

$$\Delta = 2rc_2 - (r-1){c_1}^2.$$

The well known Bogomolov inequality [1] asserts that for semi-stable bundles the discriminant is non-negative. In [2] the author of that paper posed the following problem:

Problem 1.1. Construct a sequence of μ-stable vector bundles E_m with effectively computable discriminants Δ_m such that Δ_m goes to infinity with m.

The above problem is connected with the following conjecture from [3]:

Conjecture 1.2. *Let S be a simply connected surface with trivial or ample canonical bundle. Then the Chern classes of any stable vector bundle with nontrivial moduli space obey the following improved Bogomolov inequality:*

$$\Delta \geq \frac{r^2}{12} c_2(S).$$

This conjecture is false as proved in [4,2] and [5]. In [2] and [5] the counterexamples were constructed using the method of elementary transformation on arbitrary surfaces as in the conjecture. In [4] there are two families of counterexamples: on $K3$ surfaces using the specific methods for such surfaces from [6] and on surfaces in $\mathbb{P}^3$ of degree $d \geq 7$ using Horrocks's theory of monads and methods from [7]. Also, in [8] is introduced the following.

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http://dx.doi.org/10.1016/j.geomphys.2015.10.013
0393-0440/

Definition 1.3. A sequence E_m for $m \geq 1$ of μ-stable vector bundles with the rank r_m and discriminant Δ_m tending to infinity is called a large family.

If $r_m = O(m^s)$ and $\Delta_m = O(m^t)$, the large family is of order (s, t).

For abelian varieties with cyclic Neron–Severi group the author of [8] proves the existence of large families of orders $(1, 2)$ and $(1, 3)$.

The aim of this note is to construct large families of vector bundles of various orders on arbitrary algebraic surfaces using the ideas from [9,10] and [11]. In Section 2 we describe the method of Li–Qin from [11]. Our main result concerning the construction of large families on arbitrary algebraic surfaces is Theorem 3.2 and is proved in Section 3. In particular our method shows that on any surface, the Strong Bogomolov Inequality SBI_l is false for all $l > 4$. This fact and the relation with the original Douglas–Reinbacher–Yau Conjecture is described in Section 4. The main ingredient for our examples is the Serre construction, the results are valid on any surface and the bounds are very explicit. Through the paper we shall denote by $\sim$ the equality up to terms of lower degree and by surface we always mean a smooth projective one over $\mathbb{C}$.

2. Notations and preliminaries

Let S be a smooth projective algebraic surface and L a very ample polarization on S. In [9] for the rank-2 case and in [11] for arbitrary rank, the authors use the Serre construction to obtain L-stable vector bundles with prescribed Chern classes, provided that c_2 is sufficiently great. In fact we have the following existence theorem from [11] which is a generalization of the similar, rank-2 case result, from [9]:

Theorem 2.1. *Let L be eventually rescaled such that $r \cdot L^2 > K_X \cdot L$ and*

$$\alpha = (r-1)[1 + max(p_g, h^0(S, \mathcal{O}_S(rL - c_1 + K_S))) + 4(r-1)^2 \cdot L^2] + (r-1)c_1 \cdot L - \frac{r(r-1)}{2} \cdot L^2.$$

Let $c_2 \geq \alpha$. Then there exists an L-stable rank r vector bundle E with Chern classes c_1 and c_2.

Moreover we have $h^2(X, ad(E)) = 0$.

For the convenience of the reader we recall the main steps of the proof.

– First of all, for Z a reduced 0-cycle and two line bundles L, L' on S, it is well known that there exists a locally free extension in $Ext^1(\mathcal{O}_S(L) \otimes \mathcal{I}_Z, \mathcal{O}_S(L'))$ iff Z satisfies the Cayley–Bacharach property with respect to the linear system $\mathcal{O}_S(L - L' + K_S)$.

– This fact is generalized by Li–Qin in the following:

Proposition 2.2. *Consider $r-1$ line bundles $L_1, \ldots, L_{r-1}$ and 0-cycles $Z_1, \ldots, Z_{r-1}$ on S; let $W = \oplus(\mathcal{O}_S(L_i) \otimes \mathcal{I}_{Z_i})$. Then, there is a locally free extension in $Ext^1(W, \mathcal{O}_S(L'))$ iff for any $i = 1 \ldots (r-1)$, Z_i satisfies the Cayley–Bacharach property with respect to the linear system $\mathcal{O}_S(L_i - L' + K_S)$.*

– The proposition above is used by choosing $L_i = L$ for all i's, $L' = c_1 + (1-r)L$ and a convenient 0-cycle Z of appropriate length which produces a locally free extension E with the desired Chern classes.

– The next step is the proof of the stability of E which is a consequence of some properties of the chosen 0-cycle Z.

– The fact that $h^2(X, ad(E)) = 0$, which means that the component of E in the moduli space is smooth, follows again from the properties of the 0-cycle Z using some ideas from [12].

Theorem 2.1 will be used in the next section to produce large families of various orders on arbitrary algebraic surfaces.

3. The construction of large families

Let us fix an algebraic surface S. In what follows, we shall denote by L_0 a fixed polarization on S and by $L = aL_0$ a high multiple of L_0. We shall consider the first Chern class of the form $c_1 = bL_0$ with $r \cdot a - b$ having a constant (sufficiently great) value b_0; r will be the rank of the bundle we want to obtain. Actually, in our asymptotic evaluations we shall consider that r and a vary as certain powers

$$r \sim m^s$$

and

$$a \sim m^x$$

for natural m and real positive s, x.

Remark 3.1. For $r \cdot a - b$ fixed but higher than a certain bound, the Riemann–Roch formula applied for $\mathcal{O}(rL - c_1 + K)$ gives:

$$h^0(S, \mathcal{O}_S(rL - c_1 + K_S)) = \chi(\mathcal{O}_S) + \frac{1}{2}(rL - c_1 + K_S)(rL - c_1) = constant$$

as far as $r \cdot a - b$ and consequently $rL - c_1$ are fixed.