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A theorem about energy and volume of vector fields with the "proportional volume property"

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ABSTRACT

In this paper, we define a certain *proportional volume property* for a unit vector field on a spherical domain in $\mathbb{S}^3$. We prove that the volume of these vector fields has an absolute minimum, and that this value is equal to the volume of the Hopf vector field. Some examples of such vector fields are given. We also study the minimum energy of solenoidal vector fields which coincides with a Hopf flow along the boundary of a spherical domain of $\mathbb{S}^{2k+1}$.

1. Introduction

The *volume* of a unit vector field X on a compact and oriented Riemannian n-manifold K can be defined as the volume of the submanifold $X(K)$ of the unit tangent bundle equipped with the restriction of the Sasaki metric. It is given by

$$\text{vol}(X) = \int_K \sqrt{\det(Id + (\nabla X)^T \nabla X)}\, d\nu$$

where $d\nu$ is the volume element determined by the metric and ∇ is the Levi-Civita connection. On the other side, the *energy* of a unit vector field X defined on K is given by

$$\mathcal{E}(X) = \frac{n}{2}\text{vol}(K) + \frac{1}{2}\int_K \|\nabla X\|^2\, d\nu.$$

The case where K is a submanifold with boundary of $\mathbb{S}^{2k+1}$ was treated first in [1] and the following general "boundary version" was obtained

Theorem 1 ([1]). *Let U be an open set of the $(2k+1)$-dimensional unit sphere $\mathbb{S}^{2k+1}$, let $K \subset U$ be a connected $(2k+1)$-submanifold with boundary of the sphere $\mathbb{S}^{2k+1}$ and let $\vec{v}$ be a unit vector field on U which coincides with a Hopf flow H along the boundary of K. Then*

$$\text{vol}(\vec{v}) \geq \frac{4^k}{\binom{2k}{k}}\text{vol}(K) \quad \text{and} \quad \mathcal{E}(\vec{v}) \geq \left(\frac{2k+1}{2} + \frac{k}{2k-1}\right)\text{vol}(K).$$

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If $k = 1$ we obtain that $\mathrm{vol}(\vec{v}) \geq 2\,\mathrm{vol}(K)$, the volume of the Hopf vector field. In the proof of Theorem 1 was used the following map

$$\varphi_t^{\vec{v}} : \mathbb{S}^{2k+1} \longrightarrow \mathbb{S}^{2k+1}(\sqrt{1+t^2}), \quad \varphi_t^{\vec{v}}(x) = x + t\vec{v}(x)$$

which was first used in Milnor's paper [2].

In this work we continue to study the volume and energy of vector fields defined on submanifolds with boundary $K \subset \mathbb{S}^3$. In order to find similar results, we say that a unit vector field $\vec{v}$ satisfies the "proportional volume property" if the following inequality holds for some $t > 0$

$$\frac{\mathrm{vol}(\varphi_t^{\vec{v}}(K))}{\mathrm{vol}(\mathbb{S}^3(\sqrt{1+t^2}))} \geq \frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathbb{S}^3)}.$$

This condition is obviously equivalent to

$$\mathrm{vol}(\varphi_t^{\vec{v}}(K)) \geq \mathrm{vol}(K)(1+t^2)^{3/2}.$$

We will show examples of vector fields satisfying the proportional volume property. Our goal is also to prove the following theorem.

Theorem 2. *Let K be a connected 3-submanifold with boundary of the unit sphere $\mathbb{S}^3$ and let $\vec{v}$ be a unit vector field defined on $\mathbb{S}^3$. Consider the following map $\varphi_t^{\vec{v}} : \mathbb{S}^3 \longrightarrow \mathbb{S}^3(\sqrt{1+t^2})$ given by $\varphi_t^{\vec{v}}(x) = x + t\vec{v}(x)$ and a real number $t > 0$ small enough so that the map $\varphi_t^{\vec{v}}$ is a diffeomorphism. Suppose also that the following conditions hold:*

1. $\int_{\partial K} \langle \vec{v}, \eta \rangle = 0$, *where η is the conormal vector field to ∂K.*
2. $\frac{\mathrm{vol}(\varphi_t^{\vec{v}}(K))}{\mathrm{vol}(\mathbb{S}^3(\sqrt{1+t^2}))} \geq \frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathbb{S}^3)}$.

Then $\mathrm{vol}(\vec{v}) \geq \mathrm{vol}(H)$ and $\mathcal{E}(\vec{v}) \geq \mathcal{E}(H)$, where H is the Hopf flow.

Clearly, if the vector field $\vec{v}$ is solenoidal, the condition 1 of Theorem 2 is satisfied. As a corollary of Proposition 1 in [3], there is a lower limit for the energy of solenoidal fields defined on odd-dimensional Euclidean spheres.

Theorem 3 ([3]). *The Hopf vector fields have minimum energy among all solenoidal unit vector fields on the sphere $\mathbb{S}^{2k+1}$.*

Using the same techniques of [1], we also prove the following boundary version of Theorem 3.

Theorem 4. *Let U be an open set of the $(2k+1)$-dimensional unit sphere $\mathbb{S}^{2k+1}$ and let $K \subset U$ be a connected $(2k+1)$-submanifold with boundary of the sphere $\mathbb{S}^{2k+1}$. Let $\vec{v}$ be a solenoidal unit vector field on U which coincides with a Hopf flow H along the boundary of K. Then*

$$\mathcal{E}(\vec{v}) \geq \left(\frac{2k+1}{2} + k\right) \mathrm{vol}(K) = \mathcal{E}(H).$$

2. Preliminaries

Using an orthonormal local frame $\{e_1, \ldots, e_{n-1}, e_n = \vec{v}\}$ of K, the volume of the unit vector field $\vec{v}$ is given by

$$\mathrm{vol}(\vec{v}) = \int_K \left(1 + \sum_{a=1}^{n} \|\nabla_{e_a}\vec{v}\|^2 + \sum_{a<b} \|\nabla_{e_a}\vec{v} \wedge \nabla_{e_b}\vec{v}\|^2 + \cdots + \sum_{a_1<\cdots<a_{n-1}} \left\|\nabla_{e_{a_1}}\vec{v} \wedge \cdots \wedge \nabla_{e_{a_{n-1}}}\vec{v}\right\|^2\right)^{1/2} d\nu.$$

Now, let $U \subset \mathbb{S}^{2k+1}$ be an open set of the unit sphere and let $K \subset U$ be a connected $(2k+1)$-submanifold with boundary of $\mathbb{S}^{2k+1}$. Suppose that $\vec{v}$ is a unit vector field defined on U. We also consider the map $\varphi_t^{\vec{v}} : U \longrightarrow \mathbb{S}^{2k+1}(\sqrt{1+t^2})$ given by $\varphi_t^{\vec{v}}(x) = x + t\vec{v}(x)$. We assume that $t > 0$ is small enough so that the map $\varphi_t^{\vec{v}}$ is a diffeomorphism.

In [1], the authors showed that the determinant of the Jacobian matrix of $\varphi_t^{\vec{v}}$ can be expressed in the form

$$\det(d\varphi_t^{\vec{v}}) = \sqrt{1+t^2}\left(1 + \sum_{i=1}^{2k} \sigma_i(\vec{v})t^i\right)$$

where, by definition, $h_{ij}(\vec{v}) := \langle \nabla_{e_i}\vec{v}, e_j \rangle$ (with $i, j \in \{1, \ldots, 2k\}$) and the functions σ_i are the i-symmetric functions of the h_{ij}. For instance, if $k = 1$

$$\sigma_1(\vec{v}) = h_{11}(\vec{v}) + h_{22}(\vec{v})$$
$$\sigma_2(\vec{v}) = h_{11}(\vec{v})h_{22}(\vec{v}) - h_{12}(\vec{v})h_{21}(\vec{v})$$