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The classification of algebras of level two

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ABSTRACT

This paper is devoted to the description of complex finite-dimensional algebras of level two. We obtain the classification of algebras of level two in the varieties of Jordan, Lie and associative algebras.

1. Introduction

Important subjects playing a relevant role in Mathematics and Physics are degenerations, contractions and deformations of algebras.

Degenerations of non-associative algebras were the subject of numerous papers (see for instance [1–4] and references given therein), and their research continues actively [5–7].

The general linear group $GL(V)$ over a field $\mathbb{K}$ acts on the finite-dimensional vector space $V^* \otimes V^* \otimes V$, the space of $\mathbb{K}$-algebra structures, by the change of basis. For two $\mathbb{K}$-algebra structures λ and μ we say that μ is a degeneration of λ if μ lies in the orbit closure of λ with respect to Zariski topology (it is denoted by $\mu \to \lambda$). The orbit closure problem from a geometrical point of view consists of the classification of all degenerations of a certain algebra structure of a fixed dimension. This problem also depends on a complete classification of the corresponding algebra structures. Both problems are highly complicated even in small dimensions.

It is known that closures of orbits in Zariski and standard topologies coincide in the case of an algebraically closed field of characteristic zero and as a particular case usually considered the field $\mathbb{C}$. Therefore, mainly the degenerations of complex objects are investigated.

It is well-known that there are closed relations between associative, Lie and Jordan algebras. In fact, commutator product defined on associative algebra gives us Lie algebra, while symmetrized product gives Jordan algebra. Moreover, any Lie algebra is isomorphic to a subalgebra of a certain commutator algebra. The analogue of this result is not true for Jordan algebras, that is, there are Jordan algebras which cannot be obtained from symmetrized product on associative algebras (such type of algebras are called exceptional Jordan algebras).

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The description of degenerations of dimensions less than five for complex Lie algebras and for nilpotent ones of dimensions less than seven was done in [8,9]. In the case of Jordan algebras we have the description of degenerations up to dimension four [10].

Since any n-dimensional algebra degenerates to the abelian algebra (denoted by $\mathfrak{a}_n$), the lowest edges in degenerations graph end on $\mathfrak{a}_n$. In [11] Gorbatsevich described the nearest-neighbor algebras to $\mathfrak{a}_n$ (algebras of level one) in the degeneration graphs of commutative and skew-symmetric algebras. In the work [12] it was ameliorated and correction of some non-accuracies made in [11]. Namely, a complete list of algebras level one in the variety of finite-dimensional complex algebras is obtained.

In fact, Gorbatsevich studied in [13] a very interesting notion closely related to degeneration: $\lambda \to \mu$ (algebras λ and μ not necessarily have the same dimension) if $\lambda \oplus \mathfrak{a}_k$ degenerates to $\mu \oplus \mathfrak{a}_m$ in the sense considered in this paper for some suitable $k, m \geq 0$. The corresponding first three levels of such type of degenerations are completely classified in [13].

In this paper we study the description of finite-dimensional algebras of level two over the field of complex numbers. More precisely, we obtain the classification of algebras of level two in the varieties of Jordan, Lie and associative algebras.

In the multiplication table of an algebra omitted products are assumed to be zero. Moreover, due to commutatively and anticommutatively of Jordan and Lie algebras, symmetric products for these algebras are also omitted.

2. Preliminaries

In this section we give some basic notions and concepts used through the paper.

Let λ be a n-dimensional algebra. We know that the algebra λ may be considered as an element of the affine variety $Hom(V \otimes V, V)$ via the mapping $\lambda : V \otimes V \to V$ over a field $\mathbb{K}$. The linear reductive group $GL_n(\mathbb{K})$ acts on the variety of n-dimensional algebras Alg_n via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\Big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\Big), \quad g \in GL_n(\mathbb{K}),\ \lambda \in Alg_n.$$

The orbits $Orb(-)$ under this action are the isomorphism classes of algebras. Note that solvable (respectively, nilpotent) algebras of the same dimension also form an invariant subvariety of the variety of algebras under the mentioned action.

Definition 2.1. An algebra λ is said to degenerate to an algebra μ, if $Orb(\mu)$ lies in the Zariski closure of $Orb(\lambda)$. We denote this by $\lambda \to \mu$.

The degeneration $\lambda \to \mu$ is called *trivial*, if λ is isomorphic to μ. Non-trivial degeneration $\lambda \to \mu$ is called *direct degeneration* if there is no chain of non-trivial degenerations of the form: $\lambda \to \nu \to \mu$.

Definition 2.2. The level of a n-dimensional algebra λ is the maximum length of a chain of direct degenerations, which, of course, ends with the algebra $\mathfrak{a}_n$ (the algebra with zero multiplication).

Here we give the description of the algebras of level one.

Theorem 2.3 (*[12]*)**.** *A n-dimensional* ($n \geq 3$) *algebra is algebra of level one if and only if it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
p_n^- :&\quad e_1e_i = e_i, \quad e_ie_1 = -e_i, \quad 2 \leq i \leq n;\\
n_3^- \oplus \mathfrak{a}_{n-3} :&\quad e_1e_2 = e_3, \quad e_2e_1 = -e_3;\\
\lambda_2 \oplus \mathfrak{a}_{n-2} :&\quad e_1e_1 = e_2;\\
\nu_n(\alpha) :&\quad e_1e_1 = e_1, \quad e_1e_i = \alpha e_i, \quad e_ie_1 = (1-\alpha)e_i, \quad 2 \leq i \leq n,\ \alpha \in \mathbb{C}.
\end{aligned}$$

Note that algebras $\lambda_2 \oplus \mathfrak{a}_{n-2}$ and $\nu_n\left(\frac{1}{2}\right)$ are Jordan algebras.

It is remarkable that the notion of degeneration considered in [13] is weaker than notions which are used in this paper. For instance, the levels by Gorbatsevich's work of the algebras p_n^- and $\nu_n(\alpha)$ do not equal one, because of $p_n^- \oplus \mathfrak{a}_1 \to n_3^- \oplus \mathfrak{a}_{n-2}$ and $\nu_n(\alpha) \oplus \mathfrak{a}_1 \to \lambda_2 \oplus \mathfrak{a}_{n-1}$.

It is known that any finite-dimensional associative (Jordan) algebra A is decomposed into a semidirect sum of semi-simple subalgebra A_{ss} and nilpotent radical $Rad(A)$. Moreover, an arbitrary finite-dimensional semi-simple associative (Jordan) algebra contains an identity element. Therefore, one can assume that a finite-dimensional associative (Jordan) algebra over a field $\mathbb{K}$ of $char\mathbb{K} = 0$ is either nilpotent or has an idempotent element.

One of the important results of theory of associative algebras related with idempotents is Pierce's decomposition. Let A be an associative algebra which contains an idempotent element e. Then we have decomposition

$$A = A_{1,1} \oplus A_{1,0} \oplus A_{0,1} \oplus A_{0,0}$$

with property $A_{i,j} \cdot A_{k,l} \subseteq \delta_{j,k} A_{i,l}$, where $\delta_{j,k}$ are Kronecker symbols. The subspaces $A_{j,k}$ are called Pierce's components.