Journal of Geometry and Physics 98 (2015) 21–27

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Isometric immersions of generalized Berger spheres in $\mathbb{S}^4(1)$ and $\mathbb{C}P^2(4)$

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ARTICLE INFO

Article history:
Received 8 March 2015
Accepted 21 July 2015
Available online 29 July 2015

MSC:
53A30
53C42

Keywords:
Generalized Berger sphere
Isometric immersion
Conformal immersion

ABSTRACT

In this paper, we classify the isometric immersions of generalized Berger spheres $(\mathbb{S}^3, g_{GB})$ in $\mathbb{S}^4(1)$ and $\mathbb{C}P^2(4)$ (under proper assumption in the latter case) and show the explicit expressions of g_{GB}. As an application, we obtain infinitely many generalized Berger spheres admitting conformal immersions in $\mathbb{R}^4$, which is closely related to a question of Peng and Tang (2010).

1. Introduction

In Riemannian geometry, the well-known *Berger metrics* are a 1-parameter family of metrics on the 3-sphere obtained by performing the canonical variation along fibers of a Hopf fibration. In this paper, we consider a 3-parameter family of metrics on $\mathbb{S}^3$ which contain the classical Berger metrics as special cases. Following A. Gray [1], denote by x the position vector of the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Regarding $\mathbb{R}^4$ as the set of quaternions, we obtain vector fields ix, jx, kx tangent to $\mathbb{S}^3$. Let $\omega_1, \omega_2, \omega_3$ be the 1-forms on $\mathbb{S}^3$ given by $\omega_1(X) = \langle X, ix\rangle$, $\omega_2(X) = \langle X, jx\rangle$, $\omega_3(X) = \langle X, kx\rangle$, respectively. The 3-parameter family of metrics on $\mathbb{S}^3$ are given by:

$$g_{GB} = \alpha^2\omega_1^2 + \beta^2\omega_2^2 + \gamma^2\omega_3^2,$$

where α, β, γ are positive constants. Clearly, these can be viewed as natural generalizations of the classic *Berger metrics* on $\mathbb{S}^3$. From now on, we will call $(\mathbb{S}^3, g_{GB})$ the *generalized Berger spheres.*[1] When $(\alpha, \beta, \gamma) = (\alpha, 1, 1)$, one obtains the classical Berger spheres.

The connections and curvatures of the metric g_{GB} can be computed using the Cartan structure equations

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \end{aligned}$$

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[1] Such generalized Berger spheres are also called "of Kaluza–Klein type" in [2].

http://dx.doi.org/10.1016/j.geomphys.2015.07.022
0393-0440/

together with the relations $d\omega_1 = 2\omega_2 \wedge \omega_3, d\omega_2 = 2\omega_3 \wedge \omega_1, d\omega_3 = 2\omega_1 \wedge \omega_2$. In particular, the Ricci tensor of $(\mathbb{S}^3, g_{GB})$ is given by (see [2] for instance):

$$\begin{aligned} \mathrm{Ric}_{11} &= \frac{2(\alpha^2-\beta^2+\gamma^2)(\alpha^2+\beta^2-\gamma^2)}{\alpha^2\beta^2\gamma^2}, \qquad \mathrm{Ric}_{12} = \mathrm{Ric}_{13} = 0, \\ \mathrm{Ric}_{22} &= \frac{2(\beta^2-\gamma^2+\alpha^2)(\beta^2+\gamma^2-\alpha^2)}{\alpha^2\beta^2\gamma^2}, \qquad \mathrm{Ric}_{21} = \mathrm{Ric}_{23} = 0, \\ \mathrm{Ric}_{33} &= \frac{2(\gamma^2-\alpha^2+\beta^2)(\gamma^2+\alpha^2-\beta^2)}{\alpha^2\beta^2\gamma^2}, \qquad \mathrm{Ric}_{31} = \mathrm{Ric}_{32} = 0. \end{aligned} \tag{1}$$

Several interesting geometric properties of $(\mathbb{S}^3, g_{GB})$ are listed as follows:

(i) The Ricci tensor is positive definite if and only if $\alpha^2, \beta^2, \gamma^2$ can be regarded as three edges of some triangle by (1).
(ii) $(\mathbb{S}^3, g_{GB})$ are homogeneous Riemannian manifolds with isometry group containing $Sp(2)$, see [2] and [1].
(iii) $(\mathbb{S}^3, g_{GB})$ are $\mathcal{A}$-manifolds if and only if two of α, β, γ are equal, are $\mathcal{B}$-manifolds (or Ricci-parallel) if and only if $\alpha = \beta = \gamma$, see [2] and [1]. Recall that $\mathcal{A}$-manifolds and B-manifolds, introduced by A. Gray, are two significant classes of Einstein-like Riemannian manifolds, see [3] for infinitely many inhomogeneous examples.

In the present paper, we are concerned with the isometric immersions of $(\mathbb{S}^3, g_{GB})$ in $\mathbb{S}^4(1)$ and $\mathbb{C}P^2(4)$ and the explicit expressions of the metrics. One of our main results states as follows:

Theorem 1.1. *Let $f : (\mathbb{S}^3, g_{GB}) \to \mathbb{S}^4(1)$ be an isometric immersion of a generalized Berger sphere in $\mathbb{S}^4(1)$, then it must be one of the following:*

(i) *$f(\mathbb{S}^3)$ is a geodesic sphere of $\mathbb{S}^4(1)$, and $g_{GB} = \sin^2 t(\omega_1^2 + \omega_2^2 + \omega_3^2)$, $0 < t < \frac{\pi}{2}$;*
(ii) *$f(\mathbb{S}^3)$ is a Cartan hypersurface, which can be viewed as a tube of radius t $(0 < t < \frac{\pi}{3})$ around the Veronese surface, and $g_{GB} = 16\sin^2 t\omega_1^2 + 16\sin^2(t + \frac{\pi}{3})\omega_2^2 + 16\sin^2(t + \frac{2\pi}{3})\omega_3^2$,*

where $ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ defines a metric of constant sectional curvature 1 on $\mathbb{S}^3$.

Remark 1.1. It is interesting to remark that the result above does not hold if we replace the ambient space by a higher dimensional unit-sphere. In fact, there exists a homogeneous embedding of SO(3) in $\mathbb{S}^5(1)$ by $f : \mathrm{SO}(3) \to \mathbb{S}^5(1)$, $(x, y, x \times y) \mapsto \frac{1}{\sqrt{2}}(x, y)$, which induces a 2-fold covering immersion of $\mathbb{S}^3$ in $\mathbb{S}^5(1)$. It can be verified that the induced metric on $\mathbb{S}^3$ is given by $f^*\mathrm{d}s^2 = 4\omega_1^2 + 2\omega_2^2 + 2\omega_3^2$ (cf. [2,4]), namely, $(\mathbb{S}^3, f^*\mathrm{d}s^2)$ is a standard Berger sphere.

Remark 1.2. Recall that the Veronese surface is given by

$$V : S^2(\sqrt{3}) \to S^4(1), (x, y, z) \mapsto \left(\frac{xy}{\sqrt{3}}, \frac{yz}{\sqrt{3}}, \frac{zx}{\sqrt{3}}, \frac{1}{2\sqrt{3}}(x^2 - y^2), \frac{1}{6}(x^2 + y^2 - 2z^2)\right).$$

The tube of radius $t = \frac{\pi}{6}$ around the Veronese surface gives rise to the minimal Cartan hypersurface, whose metric is $ds^2 = 4\omega_1^2 + 16\omega_2^2 + 4\omega_3^2$ as pointed out in [5] and [6]. The tube of radius $t = \frac{\pi}{3}$ gives the *opposite* Veronese surface.

Denote by $\mathbb{C}P^m(4)$ the complex projective space endowed with the Fubini–Study metric of constant holomorphic sectional curvature 4. Weinstein [7] observed that certain geodesic hyperspheres in $\mathbb{C}P^m$ equipped with the induced metrics provide counterexamples to an extension of a result of Klingenberg. Little seems to be known about the Riemannian geometry of the other homogeneous hypersurfaces in $\mathbb{C}P^m$ (see [8,9]), such as the tubes over the complex quadrics. Cecil and Ryan ([8], pp. 494, Remark 4.3) proposed a study of the intrinsic geometry of these examples. The following result can be viewed as an attempt in this direction for the complex projective plane setting.

Theorem 1.2. *Let $f : (\mathbb{S}^3, g_{GB}) \to \mathbb{C}P^2(4)$ be an isometric immersion of a generalized Berger sphere in $\mathbb{C}P^2(4)$. Suppose in addition that $f(\mathbb{S}^3)$ is a Hopf hypersurface in $\mathbb{C}P^2$. Then it must be one of the following:*

(i) *$f(\mathbb{S}^3)$ is a geodesic sphere of $\mathbb{C}P^2$, and $g_{GB} = \sin^2 t\cos^2 t\omega_1^2 + \sin^2 t(\omega_2^2 + \omega_3^2)$, $0 < t < \frac{\pi}{2}$;*
(ii) *$f(\mathbb{S}^3)$ is congruent to a tube of radius t $(0 < t < \frac{\pi}{4})$ around the complex quadric, and $g_{GB} = 4\sin^2 2t\omega_1^2 + 4\sin^2(t + \frac{\pi}{4})\omega_2^2 + 4\sin^2(t - \frac{\pi}{4})\omega_3^2$,*

where $ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ defines a metric of constant sectional curvature 1 on $\mathbb{S}^3$.

Remark 1.3. The tube of radius $t = \frac{\pi}{8}$ around the complex quadric gives rise to the unique minimal hypersurface among the one-parameter family of tubes (cf. [8]). However, only the tube of radius $t = \frac{\pi}{12}$ carries a standard Berger metric, that is, $ds^2 = \omega_1^2 + 3\omega_2^2 + \omega_3^2$.