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Cyclic homology for Hom-associative algebras

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ABSTRACT

In the present paper we investigate the noncommutative geometry of a class of algebras, called the Hom-associative algebras, whose associativity is twisted by a homomorphism. We define the Hochschild, cyclic, and periodic cyclic homology and cohomology for this class of algebras generalizing these theories from the associative to the Hom-associative setting.

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1. Introduction

Starting with a Lie algebraic approach to non-commutative geometry [1–3], the guiding motivation behind a "non-associative geometry" is to extend the non-commutative formalism of the spectral action principle [4–7] to a non-associative framework. More specifically, in an attempt to reformulate the Grand Unified Theories based on $SU(5)$, $SO(10)$ and E_6, it is pointed out in [8], and illustrated in [9], that in the ordinary approach to physics the basic input is a "symmetry group", which is associative by nature, whereas in the spectral approach it is an "algebra", which is not necessarily associative.

On the other hand, the study of the differential geometry of quantum groups [10,11] showed the lack of an associative differential algebra structure on the standard quantum groups, and hence the need for a non-associative geometry (differential geometry with non-associative coordinate algebras) for a full understanding of the geometry of quantum groups. The first step in this direction was taken in [12], generalizing the twisted cyclic cohomology of [13] to the setting of quasialgebras, in order to cover examples motivated by the Poisson geometry [14].

In the present paper we take up a similar analysis from the point of view of a different class of possibly non-associative algebras, called Hom-associative algebras, with the goal of extending the ordinary cyclic homology and cohomology of associative algebras to the non-associative setting.

The Hom-associative algebras first appeared in contexts related to physics. The study of q-deformations, based on deformed derivatives, of Heisenberg algebras, Witt and Virasoro algebras, and the quantum conformal algebras reveals a generalized Lie algebra structure in which the Jacobi identity is deformed by a linear map. These algebras first appeared in [15,16] and they were called Hom-Lie algebras. The Hom-associative algebras were first introduced in [17] and were developed in [18–23]. Briefly, a Hom-associative algebra $\mathcal{A}$ satisfies the usual algebra axioms with the associativity condition twisted by an algebra homomorphism $\alpha : \mathcal{A} \to \mathcal{A}$. More precisely we have

$$\alpha(a)(bc) = (ab)\alpha(c)$$

for all $a, b, c \in \mathcal{A}$. Thus with $\alpha = \mathrm{Id}$ we recover the associative algebras as a subclass.

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With the goal of extending the formal deformation theory (introduced in [24] for associative algebras, and in [25] for Lie algebras) to Hom-associative and Hom-Lie algebras, a cohomology theory for Hom-associative algebras was introduced in [18]. The first and the second cohomology groups of a Hom-associative algebra thus defined were adapted to the deformation theory of Hom-associative algebras, and generalized the Hochschild cohomology of an algebra with coefficients in the algebra itself.

The first and the second cohomology groups of [21] were later analyzed more conceptually in [26] where the authors defined a Hochschild cohomology for Hom-associative algebras (generalizing the ordinary Hochschild cohomology of an algebra with coefficients in itself) with a Gerstenhaber bracket endowing the differential complex with a graded Lie algebra structure.

The purpose of the present paper is to extend the usual notions of cyclic homology and cohomology for associative algebras to the setting of Hom-associative algebras. The lack of associativity, as the first obstacle on the way to this extension, is partly overcome by restricting our scope to the multiplicative Hom-associative algebras. With this multiplicativity assumption, the presence or absence of a unit plays a very important role: the classes of naturally occurring examples are very different in flavor. In particular, the multiplicative unital Hom-associative algebras are very close to being associative. On the other hand, in the absence of a unit, one cannot define the Connes boundary map B [27], hence we do not have the (b, B)-complex interpretation of cyclic (co)homology. Similarly, since we cannot define (co)degeneracy operators, we cannot use the cyclic module approach of [28] to cyclic (co)homology. As a result, we focus only on defining the cyclic (co)homology as a (co)kernel of Hochschild cohomology [27], and using the bicomplex approach of [29].

This requires a discussion of the Hochschild cohomology with coefficients. We recall that the cyclic homology of an associative algebra A is given by the coinvariants of the Hochschild homology of A with coefficients in A under the cyclic group action, whereas the cyclic cohomology of A is computed by the cyclic invariants of the Hochschild cohomology of A with coefficients in the dual space A^*. In the case of Hom-associative algebras, it is only the Hom-associative algebra $\mathcal{A}$ itself that has been considered as a coefficient space, by which a Hochschild cohomology theory was defined in [26]. In the present paper, on the other hand, we define a Hochschild homology theory that can admit the Hom-associative algebra $\mathcal{A}$ itself as coefficients, and a Hochschild cohomology theory that can admit $\mathcal{A}^*$ as a coefficient space.

A rather surprising fact in the Hom-associative setting is that for a Hom-associative algebra $\mathcal{A}$, the dual space $\mathcal{A}^*$ is not an $\mathcal{A}$-bimodule (in the sense of [21, Def. 1.5]) via the coregular action. Furthermore, modifying the coregular action by the homomorphism that twists the associativity does not fix this problem. One of the most natural options then is to impose further conditions on the Hom-associative algebra $\mathcal{A}$ so that $\mathcal{A}^*$ becomes an $\mathcal{A}$-bimodule. For instance, if $\alpha \in \mathrm{End}(\mathcal{A})$ is an element of the centroid [30], then $\mathcal{A}^*$ is an $\mathcal{A}$-bimodule. The other option is to introduce a variant of $\mathcal{A}^*$ as the coefficient space so that it becomes $\mathcal{A}^*$ in the case when $\mathcal{A}$ is associative. This can be achieved by defining

$$\mathcal{A}^\circ = \{f \in \mathcal{A}^* \mid f(x\alpha(y)) = f(\alpha(xy)) = f(\alpha(x)y)\}$$

as we discuss below. In the case of an associative algebra A, we recall from [27] that it is precisely the Hochschild cohomology $H^n(A, A^*)$ with coefficients in A^* that is equal to the space of the de Rham currents of dimension n when $A = C^\infty(M)$, the algebra of smooth functions on a compact smooth manifold M. Hence, in order to capture the correct geometric data, we unify all of these methods in a Hochschild cohomology theory for a Hom-associative algebra $\mathcal{A}$ with coefficients in a new object which we call a dual module.

The paper is organized as follows. In Section 2 we review the basics of Hom-associative algebras, in particular we study those having α in the centroid. We note for a Hom-associative algebra $\mathcal{A}$ that $\mathcal{A}^*$ is not necessarily an $\mathcal{A}$-bimodule, and thus we investigate the algebraic dual of a module over a Hom-associative algebra from the representation theory point of view. Finally, we recall the basics of Hochschild and cyclic homology and cohomology for associative algebras. In Section 3 we introduce Hochschild homology of $\mathcal{A}$ with coefficients in a bimodule V. Then for $V = \mathcal{A}$ we introduce the cyclic group action on the Hochschild complex, and define the cyclic homology of a multiplicative Hom-associative algebra $\mathcal{A}$. Finally, we use the bicomplex method to define the cyclic and the periodic homologies of $\mathcal{A}$, and we show that the equivalency of the two definitions. In Section 4 we define the Hochschild cohomology of $\mathcal{A}$ with coefficients in a new object V which we call a dual module. We define the cyclic group action on the Hochschild complex of $\mathcal{A}$, with coefficients in the dual module $\mathcal{A}^*$, and hence the cyclic cohomology of $\mathcal{A}$. Similar to the homology case, we use the bicomplex method to define the cyclic and periodic cyclic cohomologies of $\mathcal{A}$, and we show that the two definitions agree.

Notation. Throughout the paper all algebras are over a field. We reserve the font A for associative algebras whereas for Hom-associative algebras we use $\mathcal{A}$. All tensor products are over the field of the algebra in question.

2. Preliminaries

2.1. Hom-associative algebras

In this subsection we recall the definition of a Hom-associative algebra, and in addition to basic examples, we study characterization results on the unitality and the embedding properties of Hom-associative algebras.

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