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A double Poisson algebra structure on Fukaya categories

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ABSTRACT

Let M be an exact symplectic manifold with $c_1(M) = 0$. Denote by $\mathrm{Fuk}(M)$ the Fukaya category of M. We show that the dual space of the bar construction of $\mathrm{Fuk}(M)$ has a differential graded noncommutative Poisson structure. As a corollary we get a Lie algebra structure on the cyclic cohomology $\mathrm{HC}^\bullet(\mathrm{Fuk}(M))$, which is analogous to the ones discovered by Kontsevich in noncommutative symplectic geometry and by Chas and Sullivan in string topology.

1. Introduction

In this paper we construct a noncommutative Poisson structure on the Fukaya category of an exact symplectic manifold with vanishing first Chern class. Our motivation comes from the noncommutative symplectic geometry [1–5], noncommutative Poisson geometry [6–8] and string topology [9,10]. Let us start with some backgrounds.

Roughly speaking, Fukaya category is an algebraic structure arising in the study of symplectic manifolds, where the objects are Lagrangian submanifolds and the morphisms are Lagrangian intersection Floer cochain complexes. As observed by Fukaya [11], the composition of two morphisms is not associative, but associative up to homotopy. There are homotopy of homotopies, and homotopy of homotopies of homotopies, etc., forming an A_∞ category, a categorical generalization of Stasheff's A_∞ algebra.

Ever since its first appearance, Fukaya category has been a fast developing topic, and is active in, to name a few, symplectic geometry, homological and homotopical algebra, noncommutative geometry and mathematical physics. It is one of the noncommutative *symplectic* spaces in Kontsevich's homological mirror symmetry program. In fact, Kontsevich [12] and also Costello [13] conjectured that the Fukaya category of a Calabi–Yau manifold is a Calabi–Yau A_∞ category, which means there is a non-degenerate symmetric bilinear pairing of degree d on the morphism spaces

$$\langle -, - \rangle : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, A) \to k,$$

for any objects A and B, where k is the ground field of characteristic zero, such that it is cyclically invariant

$$\langle m_n(a_0, \ldots, a_{n-1}), a_n \rangle = (-1)^{n+|a_0|(|a_1|+\cdots+|a_n|)} \langle m_n(a_1, \ldots, a_n), a_0 \rangle. \qquad (*)$$

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And the famous homological mirror symmetry conjecture of Kontsevich says that, the (derived category of the) Fukaya category of a Calabi–Yau manifold should be equivalent, as Calabi–Yau categories, to the (derived category of) coherent sheaves of its mirror, and vice versa.

In general, it is very difficult to obtain a non-degenerate pairing on a Fukaya category; some partial results can be found in Fukaya [14]. On the other hand, being Calabi–Yau is very important for Fukaya categories, as they would then have very nice algebraic and geometric properties (see, for example, Kontsevich–Soibelman [15] and Costello [13]).

One nice property of a Calabi–Yau category is a Lie algebra structure on its cyclic cohomology (as shall be recalled in later sections), which is nowadays also called the *Kontsevich bracket*, and has found many applications in noncommutative symplectic/Poisson geometry, representation theory of quiver algebras and Calabi–Yau algebras. Since this Lie algebra is a main motivation of our study, we would like to say some more words about it.

In two very influential papers [1,2], Kontsevich first raised his theory of noncommutative symplectic geometry. In particular, he showed that for a noncommutative symplectic space, the noncommutative 0-forms possess a Lie algebra structure, whose homology is intimately related to the homology of some corresponding moduli space. His result was later further studied and developed by Ginzburg in [3] and Bocklandt–Le Bruyn in [4]. These authors proved that the closed path of a doubled quiver has a Lie algebra structure (the Kontsevich bracket), which is naturally mapped to the Lie algebra of Hamiltonian functions on the corresponding quiver varieties. Kontsevich's Lie algebra is first considered by Van den Bergh in [6] from the noncommutative Poisson geometry point of view. The relationship between noncommutative symplectic and noncommutative Poisson structures is also discussed in [6, Appendix].

In fact, what Van den Bergh introduced is, for a general associative algebra A, the notion of a *double Poisson bracket*. If the algebra A possesses a double Poisson bracket, then he showed that the commutator quotient space $A_\natural := A/[A, A]$ has a Lie algebra structure, where Kontsevich's Lie algebra is a special case when A is the path algebra of a doubled quiver. It turns out that Van den Bergh's double Poisson algebra is a very important case of Crawley-Boevey's noncommutative Poisson structures [7]. The study of Crawley-Boevey was motivated by his trying to find the *weakest* condition for an associative algebra A such that the moduli space of representations (representation scheme) of A admits a Poisson structure. If such a condition is fulfilled, we say A possesses a *noncommutative Poisson structure*. This idea fits very well to a guiding principle proposed by Kontsevich and Rosenberg [16], namely, for a noncommutative space, any meaningful noncommutative geometric structure (such as noncommutative symplectic and Poisson) should induce its classical counterpart on its moduli space of representations.

Now, let us go back to Fukaya categories. As we have said, it is in general very difficult to prove that a Fukaya category is indeed a Calabi–Yau category. Nevertheless, we found that Kontsevich's Lie algebra does not *a priori* assume the existence of a non-degenerate pairing, but cyclic invariance (in an appropriate sense) is essential. This is exactly the case of Fukaya categories, where the counting of the pseudo-holomorphic disks is cyclically invariant. That is to say, there is a natural Lie algebra structure on the cyclic cohomology of a Fukaya category, and such a Lie algebra is a consequence of the noncommutative Poisson structure (in the sense of Van den Bergh) on the Fukaya category, when viewing it as a noncommutative space. The following is our main theorem:

Theorem A (Theorem 17). *Let M be an exact symplectic $2d$-manifold with $c_1(M) = 0$ and possibly with contact type boundary. Denote by $\mathrm{Fuk}(M)$ the Fukaya category of M. Then the dual space of the bar construction of $\mathrm{Fuk}(M)$ has a degree $2-d$ differential graded double Poisson algebra structure in the sense of Van den Bergh.*

As a corollary (Corollary 19), the cyclic cohomology of the Fukaya category of an exact symplectic manifold with vanishing first Chern class has a degree $2 - d$ graded Lie algebra structure.

The rest of the paper is devoted to the proof of Theorem A. It is organized as follows: In Section 2 we first recall the definition of A_∞ categories and their Hochschild and cyclic (co)homologies, and then construct a double Poisson bracket on a class of A_∞ categories. In Section 3 we first briefly recall the construction of Fukaya categories and then prove Theorem A; after that, we discuss some relations of the main result to string topology, a theory developed by Chas and Sullivan [9,10]; finally, we give the detailed proof of Lemma 4 in Appendix A.

Convention. Throughout the paper, we fix a ground field k of characteristic zero. All vector spaces, their morphisms and tensor products are assumed to be over k.

2. A_∞ categories and the double Poisson bracket

In this section we first collect some necessary concepts, such as A_∞ categories and their Hochschild and cyclic (co)homologies, and then construct, for a class of A_∞ categories, a double Poisson bracket on the dual space of their bar construction.

2.1. A_∞ categories and their homologies

Definition 1 (*A_∞ Category; cf. [11,17]*). An *A_∞ category* $\mathcal{A}$ over k consists of a set of objects $\mathrm{Ob}(\mathcal{A})$, a graded k-vector space $\mathrm{Hom}(A_1, A_2)$ for each pair of objects $A_1, A_2 \in \mathrm{Ob}(\mathcal{A})$, and a sequence of multilinear maps:

$$m_n : \mathrm{Hom}(A_n, A_{n+1}) \otimes \cdots \otimes \mathrm{Hom}(A_2, A_3) \otimes \mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(A_1, A_{n+1}),$$