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A reconstruction theorem for Connes–Landi deformations of commutative spectral triples

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Dedicated to Marc Rieffel on the occasion of his 75th birthday

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ABSTRACT

We formulate and prove an extension of Connes's reconstruction theorem for commutative spectral triples to so-called Connes–Landi or isospectral deformations of commutative spectral triples along the action of a compact Abelian Lie group G, also known as toric noncommutative manifolds. In particular, we propose an abstract definition for such spectral triples, where noncommutativity is entirely governed by a deformation parameter sitting in the second group cohomology of the Pontryagin dual of G, and then show that such spectral triples are well-behaved under further Connes–Landi deformation, thereby allowing for both quantisation from and dequantisation to G-equivariant abstract commutative spectral triples. We then use a refinement of the Connes–Dubois-Violette splitting homomorphism to conclude that suitable Connes–Landi deformations of commutative spectral triples by a rational deformation parameter are almost-commutative in the general, topologically non-trivial sense.

1. Introduction

Just as the 2-torus can be deformed along its translation action on itself to obtain the noncommutative 2-tori, whether as C^*-algebras, Fréchet pre-C^*-algebras, or spectral triples, so too can more general smooth manifolds be deformed along an action of an Abelian Lie group to yield noncommutative C^*-algebras, Fréchet pre-C^*-algebras, or spectral triples. In the case of C^*-algebras or Fréchet pre-C^*-algebras, this process is Rieffel's *strict deformation quantisation* [1], whilst in the case of spectral triples and compact Abelian Lie groups, this process is Connes and Landi's *isospectral deformation* [2], which, following Yamashita [3], we call *Connes–Landi deformation*. In fact, as was first observed by Sitarz [4] and Várilly [5], Connes–Landi deformation can be viewed as none other than the adaptation to spectral triples of strict deformation quantisation along the action of a compact Abelian Lie group.

In this paper, we formulate and prove an extension of Connes's reconstruction theorem for commutative spectral triples [6] to spectral triples that, after the fact, will be Connes–Landi deformations along the action of a compact Abelian Lie group G of spectral triples of the form $(C^\infty(X), L^2(X, E), D)$, where X is a compact oriented Riemannian G-manifold and

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D is a G-invariant essentially self-adjoint Dirac-type operator on a G-equivariant Hermitian vector bundle $E \to X$, i.e., *toric noncommutative manifolds*. More precisely, we propose a suitable abstract definition of *θ-commutative spectral triples*, which closely resembles Connes's abstract definition of commutative spectral triple [7,6] except for the specification of deformation parameter θ in the second group cohomology $H^2(\widehat{G}, \mathbb{T})$ of the Pontryagin dual $\widehat{G}$ of G, which completely governs the failure of commutativity; in particular, 0-commutative spectral triples are just G-equivariant abstract commutative spectral triples. Then, we show that the Connes–Landi deformation of a θ-commutative spectral triple by $\theta' \in H^2(\widehat{G}, \mathbb{T})$ is itself $(\theta + \theta')$-commutative, thereby facilitating both quantisation from and dequantisation to G-equivariant commutative spectral triples, to which we can apply Connes's result.

In addition to extending Connes's reconstruction theorem, we also clarify a number of aspects of the general theory of Connes–Landi deformation. In particular, we use a refinement of the Connes–Dubois-Violette splitting homomorphism [8] to show that for $\theta \in H^2(\widehat{G}, \mathbb{T})$ *rational*, viz, of finite order in the group $H^2(\widehat{G}, \mathbb{T})$, sufficiently well-behaved θ-commutative spectral triples are almost-commutative in the general, topologically non-trivial sense proposed by the author [9,10] and studied by Boeijink and Van Suijlekom [11] and by Boeijink and Van den Dungen [12]. This generalises the now-folkloric example of rational noncommutative 2-tori [13].

We begin in Section 2 by reviewing Rieffel's theory of strict deformation quantisation of Fréchet pre-C^*-algebras in the case of the action of a compact Abelian Lie group [1, pp. 19–22]. In particular, we give a detailed, constructive account of the deformation of G-equivariant finitely generated projective modules over G-equivariant Fréchet pre-C^*-algebras, generalising existing results on the deformation of G-equivariant vector bundles over G-manifolds [8,14]. In fact, we obtain an explicit formula for the projection onto the deformation of a G-equivariant finitely generated projective module corresponding to a given G-invariant projection onto the original module, generalising the concrete examples studied by Connes and Landi [2, §§ II–III] and by Landi and Van Suijlekom [15].

Next, in Section 3, we recall the general theory of *Connes–Landi deformations* or *isospectral deformations*, first defined by Connes and Landi for $\mathbb{T}^2$-actions on concrete commutative spectral triples [2] and then extended by Yamashita to arbitrary $\mathbb{T}^2$-equivariant spectral triples [3]. As Sitarz [4] and Várilly [5] first showed, this amounts to a simultaneous strict deformation quantisation of the algebra $\mathcal{A}$ of a G-equivariant spectral triple $(\mathcal{A}, H, D)$ and of its G-equivariant representation on the Hilbert space H. In particular, we clarify the role of the group $H^2(\widehat{G}, \mathbb{T})$ in parametrising Connes–Landi deformations of a fixed G-equivariant spectral triple up to G-equivariant unitary equivalence, and then completely generalise the isomorphisms amongst the Morita–Rieffel equivalences of smooth noncommutative n-tori parametrised by the densely-defined $\mathrm{SO}(n, n|\mathbb{Z})$-action on the universal cover $\mathrm{Skew}(n, \mathbb{R}) \cong \mathbb{R}^{n(n-1)/2}$ of $H^2(\mathbb{Z}^n, \mathbb{T}) \cong \mathbb{T}^{n(n-1)/2}$, as introduced by Rieffel and Schwarz [16] and studied by Elliott and Li [17].

At last, in Section 4, we formulate and prove our extension of Connes's reconstruction theorem for commutative spectral triples [6, Theorem 1.1] to Connes–Landi deformations of G-equivariant commutative spectral triples. First, by analogy with Connes's abstract definition of commutative spectral triple [7,6], we propose an abstract definition of spectral triples that, after the fact, will be Connes–Landi deformations of G-equivariant concrete commutative spectral triples, where noncommutativity is entirely governed by a deformation parameter $\theta \in H^2(\widehat{G}, \mathbb{T})$ through its associated alternating bicharacter $\iota(\theta) \in \mathrm{Hom}(\wedge^2 \widehat{G}, \mathbb{T})$.

Definition 1.1. Let $(\mathcal{A}, H, D)$ be a G-equivariant regular spectral triple, let $\theta \in H^2(\widehat{G}, \mathbb{T})$, and let $p \in \mathbb{N}$. We shall call $(\mathcal{A}, H, D)$ a p-dimensional *θ-commutative* spectral triple if the following conditions all hold:

(0) *Order zero*: The algebra $\mathcal{A}$ is *θ-commutative*, viz,

$$\forall \mathbf{x}, \mathbf{y} \in \widehat{G}, \ \forall a_{\mathbf{x}} \in \mathcal{A}_{\mathbf{x}}, \ \forall b_{\mathbf{y}} \in \mathcal{A}_{\mathbf{y}}, \quad b_{\mathbf{y}} a_{\mathbf{x}} = e(\iota(\theta)(\mathbf{x}, \mathbf{y}))\, a_{\mathbf{x}} b_{\mathbf{y}},$$

so that the G-equivariant $*$-representation $L : \mathcal{A} \to B(H)$ of $\mathcal{A}$ can be deformed to a G-equivariant $*$-homomorphism $R : \mathcal{A}^{\mathrm{op}} \to B(H)$, such that for all $a, b \in \mathcal{A}$, $[L(a), R(b)] = 0$.

(1) *Dimension*: The eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $(D^2 + 1)^{-1/2}$, counted with multiplicity and arranged in decreasing order, satisfy $\lambda_n = O(n^{-1/p})$ as $n \to +\infty$.

(2) *Order one*: For all $a, b \in \mathcal{A}$, $[[D, L(a)], R(b)] = 0$.

(3) *Orientability*: Define $\varepsilon_\theta : \mathcal{A}^{\otimes(p+1)} \to \mathcal{A}^{\otimes(p+1)}$ by

$$\varepsilon_\theta(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := \frac{1}{p!} \sum_{\pi \in S_p} \exp\left(2\pi i \sum_{\substack{i<j \\ \pi(i)>\pi(j)}} \iota(\theta)(\mathbf{x}_{\pi(i)}, \mathbf{x}_{\pi(j)}) \right) (-1)^\pi a_0 \otimes a_{\pi(1)} \otimes \cdots \otimes a_{\pi(p)}$$

for isotypic elements $a_0 \in \mathcal{A}_{\mathbf{x}_0}, \ldots, a_p \in \mathcal{A}_{\mathbf{x}_p}$ of $\mathcal{A}$, and say that $\mathbf{c} \in \mathcal{A}^{\otimes(p+1)}$ is *θ-antisymmetric* if $\varepsilon_\theta(\mathbf{c}) = \mathbf{c}$. Define $\pi_D : \mathcal{A}^{\otimes(p+1)} \to B(H)$ by

$$\forall a_0, \ a_1, \ldots, a_p \in \mathcal{A}, \quad \pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := L(a_0)[D, L(a_1)] \cdots [D, L(a_p)].$$

There exists a G-invariant θ-antisymmetric $\mathbf{c} \in \mathcal{A}^{\otimes(p+1)}$, such that $\chi := \pi_D(\mathbf{c})$ is a self-adjoint unitary, satisfying

$$\forall a \in \mathcal{A}, \quad L(a)\chi = \chi L(a), \qquad [D, L(a)]\chi = (-1)^{p+1} \chi [D, L(a)].$$