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Equilibria of point charges on convex curves

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ABSTRACT

We study the equilibrium positions of three points on a convex curve under influence of the Coulomb potential. We identify these positions as *orthotripods*, three points on the curve having concurrent normals. This relates the equilibrium positions to the caustic (evolute) of the curve. The concurrent normals can only meet in the *core* of the caustic, which is contained in the interior of the caustic. Moreover, we give a geometric condition for three points in equilibrium with positive charges only. For the ellipse we show that the space of orthotripods is homeomorphic to a 2-dimensional bounded cylinder.

1. Introduction

The electrostatic (Coulomb) potential of a system of point charges confined to a certain domain in Euclidean plane has been studied in a number of papers in connection with various problems and models of mathematical physics [1–3]. Analogous problems for the gravitational potential of point masses have also been discussed in the literature [1–3].

Several issues considered in the mentioned papers were connected with studying equilibria configurations of a system of point charges. As usual, a configuration P of several points on a curve X is called an equilibrium for a system of charges Q if the system is 'at rest' at this configuration. In particular, equilibria of three point charges on an ellipse have been discussed in some detail in [1].

In this paper, we deal with a specific problem related to our previous research of charged polygonal linkages [4]. Namely, given a collection of $n \geq 3$ points P on a given (fixed) closed curve X, we wish to investigate if this collection of points is an equilibrium of E_q for a certain system of point charges q. If this is possible, any such collection of charges q will be called *balancing charges* for P. We also want to investigate if this can be done with positive charges only. An analogous definition is meaningful and interesting for several other potentials of point interactions, e.g., for gravitational potential or logarithmic potential in the plane and more general central forces.

In the sequel we deal mostly with the (electrostatic) Coulomb potential E_q and the E_q-equilibrium problem for triples of points on a closed curve in the plane. We reveal that this problem is closely related to certain geometric issues concerned

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with the concept of *caustic* (evolute) of a plane curve and present several results along these lines. In particular, we give a geometric criterion for balancing three points on an arbitrary closed curve (Theorem 1) and present a condition for balancing with positive charges (Proposition 2). We give a description of the set of triples which can be balanced on an ellipse, and also a description of the set of triples on an ellipse which can be balanced by positive charges. Some related results and arising research perspectives are discussed in the last section of the paper.

An important observation is that Coulomb forces can be replaced by Hooke forces produced by (either compressed or extended) connecting springs, or by any other *central forces*.

2. Electrostatic equilibrium of points on closed curve

2.1. Condition for triples in equilibrium

We consider a collection $P_1, \dots, P_n$ of distinct points on a smooth curve X in the plane, together with charges $q_1, \dots, q_n$. We do not require in this section that the curve is convex. The Coulomb potential of these point charges is given by

$$E_q = -\sum_{i<j} \frac{q_i q_j}{d_{ij}},$$

where $p_i = \overrightarrow{OP_i}$, and $d_{ij} = \|p_i - p_j\|$. The Coulomb forces between these points are given by

$$F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j).$$

Let $F_i = \sum_{j \neq i} F_{ji}$ be the resultant of these forces at P_i. Let T_i be the tangent vector to the curve X at the point P_i.

Definition 1. A collection of points on a curve X charged by $q = (q_1, \dots, q_n) \neq (0, \dots, 0)$ is called an E_q-equilibrium (or is E_q-balanced) if, at every point P_i of the collection, the resultant of the forces is orthogonal to T_i:

$$\langle F_i, T_i \rangle = \sum_{j \neq i} \frac{q_i q_j}{d_{ij}^3} \langle p_i - p_j, T_i \rangle = 0 \quad \forall i.$$

In this situation we say that the charges q are balancing for $P_1, \dots, P_n$.

Notice that E_q-equilibria correspond to the critical/stationary points of the potential E_q.

Whenever $q_i = 0$ for one of the charges, the system reduces to a system with one point less (the removed point can be on an arbitrary place). If none of the charges is zero then the equilibrium condition implies a system of linear equations for the values of balancing charges.

In the special case of two points we have two equations. Non-zero solutions $q = (q_1, q_2)$ only occur if P_1P_2 is orthogonal to both T_1 and T_2. So P_1P_2 is a double normal of the curve.

The main situation of our study is three points on a curve. In this case we have the matrix equation:

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0, \quad \text{where } a_{ij} = \frac{\langle p_i - p_j, T_i \rangle}{d_{ij}^3}.$$

If the rank of this system is 3 then its solution is only the triple $(0, 0, 0)$ so a genuine (non-trivial) equilibrium is impossible. Thus non-trivial stationary charges may only exist if the rank of this system does not exceed two.

Definition 2. Points P_1, P_2, P_3 satisfy the corank 1 condition if the rank of the matrix

$$(a_{ij}) = \left(\frac{\langle p_i - p_j, T_i \rangle}{d_{ij}^3} \right)$$

is less than or equal to 2.

If the rank of the matrix (a_{ij}) is equal to 2, then the matrix equation has a one-dimensional solution space and therefore defines a unique point $[q_1 : q_2 : q_3]$ in $\mathbb{P}^2$. In case where the rank of this matrix is 1 it follows that the points P_1, P_2, P_3 are collinear and at least one of P_iP_j is a double normal. This case cannot occur if the curve is convex.

Proposition 1. *Points P_1, P_2, P_3 satisfy the corank 1 condition if and only if the three normals at these points are concurrent, that is, have a common point (see Fig. 1).*