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Infinitesimal deformations of filiform Lie algebras of order 3

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ABSTRACT

The **Lie algebras of order** F have important applications for the fractional supersymmetry, and on the other hand the **filiform** Lie (super)algebras have very important properties into the Lie Theory. Thus, the aim of this work is to study **filiform Lie algebras of order** F which were introduced in Navarro (2014). In this work we obtain new families of filiform Lie algebras of order 3, in which the complexity of the problem rises considerably respecting to the cases considered in Navarro (2014).

1. Introduction

The Lie theory has very important applications in the study of symmetries in Physics. Nowadays, some generalizations of the lie (super)algebras that have been used thanks to their physics applications are color Lie superalgebras [1–4] and Lie algebras of order F [5–7]. In particular, the Lie algebras of order F were considered to implement non-trivial extensions of the Poincaré symmetry different from the usual supersymmetric extension. The Lie algebras of order F can be considered as the algebraic structure associated to fractional supersymmetry [8–11].

On the other hand the concept of "filiform", into the Lie theory, was introduced by Vergne [12]. Filiform Lie algebras have important properties, in particular all of them can be obtained by using a deformation of the model filiform Lie algebra. This result has been generalized for Lie superalgebras [13–15] and [16].

Thus, we center our present study in Lie algebras of order F and more precisely in "filiform Lie algebras of order F", which were introduced in [17] as a possible generalization of the filiform Lie (super)algebras into the theory of Lie algebras of order F. We continue with the study started in [17] obtaining thus, families of filiform elementary Lie algebras of order 3 by using infinitesimal deformations. It can be observed that the complexity of the problem rises considerably when the dimension rises (Theorems 4 and 5).

We assume that the reader is familiar with the standard Lie theory and that all the vector spaces considered through the work are $\mathbb{C}$-vector spaces with finite dimension.

2. Preliminaries

We can define a Lie superalgebra (see [18] and [19]) as a $\mathbb{Z}_2$−graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to a bracket product $[\ ,\ ]$, i.e. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta(mod 2)} \forall \alpha, \beta \in \mathbb{Z}_2$, that verifies two properties: the graded skew-symmetry $([X, Y] =$

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$-(-1)^{\alpha\cdot\beta}[Y,X]$) and the graded Jacobi identity ($(-1)^{\gamma\cdot\alpha}[X,[Y,Z]] + (-1)^{\alpha\cdot\beta}[Y,[Z,X]] + (-1)^{\beta\cdot\gamma}[Z,[X,Y]] = 0$) with $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$.

It can be observed that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, we have in particular that $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ has structure of $\mathfrak{g}_0-$module.

Now, we recall some basic results of Lie algebras of order F introduced in [5,6] and [7].

Definition 2.1 ([20]). Let $F \in \mathbb{N}^*$. A $\mathbb{Z}_F$-graded $\mathbb{C}$-vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex **Lie algebra of order** F if the following hold:

(1) $\mathfrak{g}_0$ is a complex Lie algebra.
(2) For all $i = 1, \ldots, F-1$, $\mathfrak{g}_i$ is a representation of $\mathfrak{g}_0$. If $X \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_i$, then $[X, Y]$ denotes the action of $X \in \mathfrak{g}_0$ on $Y \in \mathfrak{g}_i$ for all $i = 1, \ldots, F-1$.
(3) For all $i = 1, \ldots, F-1$, there exists an F-Linear, $\mathfrak{g}_0$-equivariant map, $\{\cdots\} : \mathcal{S}^F(\mathfrak{g}_i) \longrightarrow \mathfrak{g}_0$, where $\mathcal{S}^F(\mathfrak{g}_i)$ denotes the F-fold symmetric product of $\mathfrak{g}_i$.
(4) For all $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_k$, the following "Jacobi identities" hold:

$$[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] = 0. \tag{2.1.1}$$
$$[[X_1, X_2], Y_3] + [[X_2, Y_3], X_1] + [[Y_3, X_1], X_2] = 0. \tag{2.1.2}$$
$$[X, \{Y_1, \ldots, Y_F\}] = \{[X, Y_1], \ldots, Y_F\} + \cdots + \{Y_1, \ldots, [X, Y_F]\}. \tag{2.1.3}$$
$$\sum_{j=1}^{F+1}[Y_j, \{Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{F+1}\}] = 0. \tag{2.1.4}$$

As a Lie algebra of order 1 is exactly a Lie algebra and a Lie algebra of order 2 is a Lie superalgebra, then we can consider that the Lie algebras of order 3 are generalization of the Lie (super)algebras.

Proposition 2.2 ([20]). *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F, with $F > 1$. For any $i = 1, \ldots, F-1$, the subspaces $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ inherits the structure of a Lie algebra of order F. We call these type of algebras **elementary Lie algebras of order** F.*

We will consider in our study elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, it can be found examples of elementary Lie algebras of order 3 in [5].

Prior to give the definition of "*filiform Lie algebras of order* 3", it is necessary to know the concept "*filiform module*".

Definition 2.3 ([17]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. $\mathfrak{g}_i$ is called a $\mathfrak{g}_0$-**filiform module** if there exists a decreasing subsequence of vector subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m-1, \ldots, 0$, respectively, $m > 0$, and such that $[\mathfrak{g}_0, V_{i+1}] = V_i$.

Definition 2.4 ([17]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F. Then $\mathfrak{g}$ is a **filiform Lie algebra of order** F if the following conditions hold:

(1) $\mathfrak{g}_0$ is a filiform Lie algebra.
(2) $\mathfrak{g}_i$ has structure of $\mathfrak{g}_0$-filiform module, for all i, $1 \le i \le F-1$.

If we consider an homogeneous basis of a Lie algebra of order 3, then the Lie algebra of order 3 will be completely determined by its structure constants. These structure constants verify the polynomial equations that come from the Jacobi identity. All these equations endow to the Lie algebras of order 3 of structure of algebraic variety, called $\mathcal{L}_{n,m,p}$. The subset composed of all filiform Lie algebras of order 3 will be denoted by $\mathcal{F}_{n,m,p}$.

We consider the multiplication of the Lie algebra of order 3 $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as the linear map ψ

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) : (\mathfrak{g}_0 \wedge \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_1) \oplus (\mathfrak{g}_0 \wedge \mathfrak{g}_2) \oplus S^3(\mathfrak{g}_1) \oplus S^3(\mathfrak{g}_2) \longrightarrow \mathfrak{g} \quad \text{where,}$$

$\psi_1 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$, $\psi_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$, $\psi_3 : \mathfrak{g}_0 \wedge \mathfrak{g}_2 \longrightarrow \mathfrak{g}_2$, $\psi_4 : S^3(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0$, and $\psi_5 : S^3(\mathfrak{g}_2) \longrightarrow \mathfrak{g}_0$. As a rule ψ_1, ψ_2 and ψ_3 are represented by [,], and ψ_4 and ψ_5 by {, , }. We consider the action of the group $GL(n+1, m, p) \cong GL(n+1) \times GL(m) \times GL(p)$ on $\mathcal{L}_{n,m,p}$ (see [17]) and we denote by $\mathcal{O}_\psi$ the orbit of $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$ with respect to this action. Thus, the algebraic variety $\mathcal{L}_{n,m,p}$ is fibered by these orbits and the quotient set is the set of isomorphism classes of $(n+1+m+p)$-dimensional Lie algebras of order 3.

At this point, it is convenient to find a suitable basis, called adapted basis. This is an open problem in general, but it has been solved in some cases [1,17].