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# Slant geometry on spacelike submanifolds of codimension two in Lorentz–Minkowski space

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ABSTRACT

In this study, we construct one-parameter families of new extrinsic differential geometries on spacelike submanifolds of codimension two in Lorentz–Minkowski space. Moreover, we give our previous results as special cases of these spacelike submanifolds of codimension two. Furthermore, we investigate spacelike curves in Lorentz–Minkowski 3-space from different viewpoints as another special case.



## 1. Introduction

The analysis of differential geometries of submanifolds in Lorentz–Minkowski space is of great interest in relativity theory. In this regard, singularity theory techniques have been used recently to study differential geometries of submanifolds in Lorentz–Minkowski space [1–20].

The differential geometry of hypersurfaces in hyperbolic space (respectively, spacelike hypersurfaces in de Sitter space) was investigated in [7] (respectively, in [18]). In addition, the Legendrian dualities between the pseudo-spheres in Lorentz–Minkowski space were shown in [3]. As an application of these Legendrian dualities, the differential geometry of spacelike hypersurfaces in the lightcone was constructed in [3]. Moreover, these Legendrian dualities were extended in [21] for one-parameter families depending on a parameter $\phi \in [0, \pi/2]$ of the pseudo-spheres in Lorentz–Minkowski space. As applications of these extended Legendrian dualities, one-parameter families depending on $\phi$ for differential geometries on spacelike hypersurfaces in hyperbolic space, de Sitter space and the lightcone were established in [1,17,21]. Thus, it was shown that the results in [7] and [18] (respectively, in [3]) are special cases of the results in [1] (respectively, in [17]).

Let us explain our original motivation for the study of slant geometry. Over the past 15 years, the first author and his collaborators have constructed a new geometry called *horospherical geometry* in hyperbolic space (see [5,7,8,11,12,15]). Traditionally, there is another geometry which is the non-Euclidean geometry of Gauss–Bolyai–Lobachevski (i.e., hyperbolic

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geometry) in hyperbolic space. Let us explain both of these geometries when the dimension is two. We consider the Poincaré disk model $D^2$ of the hyperbolic plane, which is an open unit disk in the $(x, y)$ plane, with the Riemannian metric $ds^2 = 4(dx^2 + dy^2)/(1 - x^2 - y^2)^2$. This is conformally equivalent to the Euclidean plane, so a circle in the Poincaré disk is also a circle in the Euclidean plane. A geodesic in the Poincaré disk is a Euclidean circle which is perpendicular to the ideal boundary (i.e., unit circle). If we adopt geodesics as lines in the Poincaré disk, then we have the model of hyperbolic geometry. There is also another class of curves in the Poincaré disk which has an analogous property with lines in the Euclidean plane. A *horocycle* is a Euclidean circle that is tangent to the ideal boundary. We note that a line in the Euclidean plane can be considered as the limit of the circles when the radius tends to infinity. In the same manner, a horocycle is also a curve that can be considered as the limit of the circles in the Poincaré disk when the radius tends to infinity. Hence, horocycles are also an analogous notion of lines. If we adopt horocycles as lines, what kind of geometry do we obtain? We say that two horocycles are *parallel* if they have a common tangent point at the ideal boundary. The parallel axiom is satisfied under this definition. However, for any two points in the Poincaré disk, two horocycles always exist passing through these points such that the first axiom of Euclidean geometry is not satisfied. In the case of general dimensions, this geometry is referred to as *horospherical geometry*. In addition, there is another type of curve in the Poincaré disk with similar properties to Euclidean lines. An *equidistant curve* is a circle whose intersection with the ideal boundary consists of two points. In general, the angle between an equidistant curve and the ideal boundary is $\phi \in (0, \pi/2]$. Thus, we emphasize that a geodesic is an equidistant curve with $\phi = \pi/2$. However, a horocycle is not an equidistant curve, but instead it is a circle with $\phi = 0$. We refer to the geometry where $\phi = \pi/2$ as *vertical geometry* and the geometry where $\phi = 0$ as *horizontal geometry*. We also refer to the family of geometries depending on $\phi$ as *slant geometry* (see [1,17,21] for further details).

In this study, we deal with some local properties of slant geometry on spacelike submanifolds of codimension two in Lorentz–Minkowski space. Thus, we generalize some of the results obtained in [13]. Moreover, we interpret our previous results given in [1,17,21] as special cases of our results in the present study. Furthermore, we investigate spacelike curves in Lorentz–Minkowski 3-space from different viewpoints as another special case.

Throughout this study, we assume that all of the maps and manifolds are of class $C^\infty$ unless stated otherwise.

## 2. Basic notions

In this section, we first give some basic notions related to Lorentz–Minkowski space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ i = 0, 1, \dots, n\}$ be an $(n+1)$-dimensional real vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo-scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz–Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. A vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0, = 0$ or $< 0$, respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$ (cf. [22]).

For any vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n \in \mathbb{R}^{n+1}_1$, a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is defined by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ x_0^1 & x_1^1 & \cdots & x_n^1 \\ x_0^2 & x_1^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, \dots, x_n^i)$ $(i = 1, \dots, n)$. It is clear that

$$\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n\rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n),$$

so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n$ is pseudo-orthogonal to any $\boldsymbol{x}_i$.

Given $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and $c \in \mathbb{R}$, a *hyperplane* $HP(\boldsymbol{v}, c)$ *with pseudo-normal* $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v}\rangle = c\}.$$

It is said to be *spacelike*, *timelike* or *lightlike* provided that $\boldsymbol{v}$ is timelike, spacelike, or lightlike, respectively.

In $\mathbb{R}^{n+1}_1$, there are three types of pseudo-spheres, called *Hyperbolic n-space with center* $\boldsymbol{a}$ *and radius* $r$, *de Sitter n-space with center* $\boldsymbol{a}$ *and radius* $r$ and the *closed lightcone with center* $\boldsymbol{a}$ *and radius 0*, which are defined by

$$H^n(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}\rangle = -r^2\},$$
$$S^n_1(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}\rangle = r^2\}$$

and

$$LC_{\boldsymbol{a}} = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a}\rangle = 0\},$$

respectively, where $r \in \mathbb{R} \setminus \{0\}$. If the center $\boldsymbol{a}$ is $\boldsymbol{0}$, then we denote these pseudo-spheres by

$$H^n(-r^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -r^2\},$$
$$S^n_1(r^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = r^2\}$$