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A characterization of hyperbolic caps in the steady state space

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ABSTRACT

We obtain two characterizations of planar discs and hyperbolic caps of the steady state space in the family of compact spacelike surfaces with constant mean curvature: (1) they are the only ones spanning a circle; (2) they are the only capillary surfaces with boundary in a slice. We also obtain some results of symmetry when the boundary is a simple closed curve contained in a slice.

1. Introduction and statement of results

The steady state space $\mathcal{H}^3$ is a model for the universe proposed by Bondi and Gold [1] and Hoyle [2] which is homogeneous and isotropic, that is, it looks the same not only at all points and in all directions, but also at all times ([3, Sect. 5.2]. In differential geometry, the steady state space gained in interest after the work of Montiel [4] and a number of works have focused to characterize particular examples of spacelike surface of constant mean curvature (cmc surface in short). Without to be a complete list, we refer: [5–10].

We denote by $\mathbb{R}^4_1$ the 4-dimensional Minkowski space, that is, the real vector space $\mathbb{R}^4$ endowed with the Lorentz metric $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$, where $x = (x_1, x_2, x_3, x_4)$ are the canonical coordinates of $\mathbb{R}^4$. Then the de Sitter space is $\mathbb{S}^3_1 = \{x \in \mathbb{R}^4_1 : \langle x, x\rangle = 1\}$ with the induced metric from $\mathbb{R}^4_1$. Let $a \in \mathbb{R}^4_1$ be a non-zero null vector in the past half of the null cone, that is, $\langle a, a\rangle = 0$ and $\langle a, (0, 0, 0, 1)\rangle > 0$. The steady state space $\mathcal{H}^3$ is the open region of $\mathbb{S}^3_1$ defined by

$$\mathcal{H}^3 = \{p \in \mathbb{S}^3_1 : \langle p, a\rangle > 0\}.$$

This space is foliated by a uniparametric family of umbilical spacelike surfaces $L_\tau = \{p \in \mathbb{S}^3_1 : \langle p, a\rangle = \tau\}$ for each $\tau > 0$. Each leaf L_τ is called a slice of $\mathcal{H}^3$ and is isometric to the Euclidean plane.

In this paper we study compact spacelike cmc surfaces Σ in $\mathcal{H}^3$ spanning a closed curve Γ. Our interest is how the geometry of the boundary Γ determines the shape of Σ. For example, under what conditions a cmc surface Σ inherits the symmetries of its boundary Γ? Indeed, if $\Psi : \mathcal{H}^3 \to \mathcal{H}^3$ is an isometry and Γ is invariant by Ψ, i.e, $\Psi(\Gamma) \subset \Gamma$, does it hold $\Psi(\Sigma) \subset \Sigma$? The simplest case of boundary is a circle. A *circle* Γ is the set of points in a slice that are equidistant from a given point called the centre of the circle. Besides the domain bounded by Γ in the slice (a planar disc), there exist other compact spacelike cmc surfaces spanning Γ, which are pieces of hyperbolic planes. A hyperbolic plane of $\mathcal{H}^3$ meets a slice in a circle Γ which separates the hyperbolic plane in two connected components. In the case that one of such components

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is compact, it is called a hyperbolic cap. Then we prove (Theorem 15):

In the steady state space, planar discs and hyperbolic caps are the only compact spacelike cmc surfaces spanning a circle.

We point out that in [11] some estimates were given between the radius of a circle and the value of the mean curvature. A result with the same flavour appears in [12] proving in Lorentz–Minkowski space that planar discs and hyperbolic caps are the only compact spacelike cmc surfaces bounded by a circle. For other ambient spaces, as for example, Euclidean space and hyperbolic space, the knowledge of the possible configurations of cmc surfaces spanning a circle is far to be completely known. We refer the reader to [13] for more details. If in the above result we have fixed the boundary of the curve, we obtain another characterization when we impose a condition of Neumann type (or free boundary problem). Exactly we consider compact spacelike surfaces whose boundary lies in a slice but we only know that the contact angle is constant along the boundary. Then we prove (Theorem 25):

In the steady state space, planar domains and hyperbolic caps are the only compact spacelike cmc surfaces that meet a slice with constant angle along its boundary.

In Euclidean space this result is proved by graphs in the classical paper [14], proving that planar domains and spherical caps are the only cmc graphs whose boundary is a planar curve and the contact angle between the surface and the plane is constant. Using the same ideas, one can adapt the Alexandrov reflection method for embedded surfaces that lie in one side of the boundary plane [13].

Our paper is organized as follows. Once fixed the notation and some basics of $\mathcal{H}^3$, in Section 3 we give results on uniqueness of cmc surfaces with boundary in an umbilical surface of $\mathcal{H}^3$. Section 4 is devoted to prove non-existence of cmc graphs on domains of a slice if this domain is sufficiently big in relation with the value of the mean curvature. In Section 5 we prove the first characterization of planar discs and hyperbolic caps. We do two proofs of this result, firstly using a type of uniqueness of cmc surfaces with the same boundary and mean curvature and secondly, we apply the Alexandrov reflection principle [15] and we study the mean curvature equation for rotational surfaces. Finally, in Section 6 we employ again the Alexandrov reflection method to obtain some results that establish that the surface inherits the symmetries of its boundary. Here we give the second characterization of planar domains and hyperbolic caps as the only capillary surfaces with boundary in a slice. We also consider the problem of cmc surfaces spanning two circles contained in two slices. A final remark: all results in this paper hold for hypersurfaces in steady state n-space; for simplicity and illustrate the proofs, we only consider the case $n = 3$.

2. Preliminaries

Let Σ be a connected surface. A smooth immersion $\psi : \Sigma \to \mathcal{H}^3$ is said to be a spacelike surface if the induced metric via ψ is a Riemannian metric on Σ. A spacelike surface is always orientable because the causal character of the ambient space allows to choose a unique unit timelike normal vector field N globally defined on Σ which is future-directed. If ∇ stands for the Levi-Civita connection on Σ, the mean curvature H is defined as

$$H = -\frac{\mathrm{tr}(A)}{2} = -\frac{\kappa_1 + \kappa_2}{2}, \tag{1}$$

where A is the corresponding shape operator and κ_i are the principal curvatures of ψ. When H is constant we say that Σ is a cmc surface or H-surface if we want to emphasize the value of the mean curvature. We point out the choice of the minus sign in (1) is opposite to [4]: see [5,9]. Following this convention, the umbilical surfaces L_τ have constant mean curvature $H = -1$ with respect to the future unit normal vector.

Let $\Gamma \subset \mathcal{H}^3$ be a smooth closed curve. Given a surface Σ, we say that an immersion $\psi : \Sigma \to \mathcal{H}^3$ has Γ as boundary if $\psi_{|\partial\Sigma}$ is a diffeomorphism between $\partial\Sigma$ and Γ. We also briefly saying that Σ is bounded by Γ or that Σ spans Γ. Recall that if Σ is a compact spacelike surface, then Σ cannot be closed, in particular, $\partial\Sigma \neq \emptyset$. A necessary condition to Γ is the boundary of a compact spacelike surface of $\mathcal{H}^3$ is to be a spacelike curve.

In this paper, we shall use the upper half-space model $(\mathbb{R}^3_+, g)$ of $\mathcal{H}^3$ where $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_+$ is the upper half-space of vector space $\mathbb{R}^3$ endowed with metric

$$g_{(x_1,x_2,x_3)} = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 - dx_3^2), \tag{2}$$

and (x_1, x_2, x_3) are the canonical coordinates of $\mathbb{R}^3$. We will also use the terminology of horizontal and vertical in the Euclidean sense considering the target ambient space $\mathbb{R}^3_+$. In view of the metric (2), taking the Lorentz–Minkowski space $\mathbb{R}^3_1 = (\mathbb{R}^3, \langle,\rangle = dx_1^2 + dx_2^2 - dx_3^2)$ and the subset $\mathbb{R}^3_+ \subset \mathbb{R}^3$ with the induced metric, the model $(\mathbb{R}^3_+, g)$ of $\mathcal{H}^3$ is the Lorentzian analogue to the upper half-space model of hyperbolic space. We will denote by $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ the *future infinity* $\mathcal{J}^+$ [3]. We say that Γ is the asymptotic future boundary of a surface $\Sigma \subset \mathcal{H}^3$ when $\Gamma = \overline{\Sigma} \cap \mathcal{J}^+$ and the closure is taken in $x_3 \geq 0$. Recall that there exists an explicit isometry Φ between the model of $\mathcal{H}^3$ as subset of $\mathbb{S}^3_1$ and the upper half-space model and that this isometry inverts the orientation [4].

As the metric of $\mathcal{H}^3$ is conformal to the Minkowski metric, the causal character of $\mathcal{H}^3$ is the same than the upper half-space viewed as an open set of the Minkowski space $\mathbb{R}^3_1$.