Journal of Geometry and Physics 98 (2015) 285–291

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On exact solutions of Coulomb equation of motion of planar charges

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ARTICLE INFO

Article history:
Received 20 January 2014
Received in revised form 10 July 2015
Accepted 19 August 2015
Available online 28 August 2015

Keywords:
Coulomb dynamics
Keplerian motion
Regular polygon

ABSTRACT

For $n \geq 3$ point planar charges $e_j < 0, j = 1, \dots, n-1, e_n > 0$ exact Lagrange-type solutions of their Coulomb equation of motion are found. For $n > 3$ all the negative charges are identical and their masses are equal. These solutions describe a motion of the negative charges along keplerian orbits around the immobile positive charge in such a way that their coordinates coincide with vertices of a regular polygon. It is established that there exist equilibrium configurations such that the equal negative charges are located at the vertices of regular polygons centered at the positive charge. It is shown that there are no Lagrange-type triangular solutions for Coulomb equation of motion of three charges. The rectilinear Lagrange-type solution is shown to exist for it.

1. Introduction

The Maxwell–Lorentz (ML) equation of motion of arbitrary number of point charges, discussed in [1], is basic for plasma and quantum physics. When charges do not emit radiation this equation is reduced to the Coulomb or Darwin equations, derived in [1,2], which are difficult to solve (as the ML equation) due to their singular character. The important task of mathematics is a construction of solutions of all these equation on the infinite time interval. As one considers Coulomb systems then one tries to adapt ideas from the celestial mechanics exposed in a remarkable way in [3]. We showed in [4,5] that it is possible to prove the existence of solutions for the Coulomb and Darwin equations on a finite time interval excluding collisions between charges generalizing bounds from the celestial mechanics. In both cases the solutions are holomorphic functions of time in a disc centered at the origin. In the Coulomb case we established in [4] that their closest to the origin real singularity is generated by a collision. How one can avoid collisions in charge dynamics and construct it on the infinite time interval? We give positive answer to this question producing exact solutions for the special array of n charges $e_j < 0, j = 1, \dots, n-1, e_n > 0$. For $n > 3$ all masses of negative charges are equal, $e_j = -e_0, j = 1, \dots, n-1$ and $e_n > e_n^0$. For $n = 3$ these conditions are also allowed but in general the charges and their masses may be different provided $e_3 > -e_j, j = 1, 2$. We use the following remarkable fact: our systems for $n > 3$ with the positive charge e_n^0 have the equilibrium configurations for which the negative charges are located at the vertices of regular polygons centered at the origin where the positive charge is placed if

$$e_n^0 = 2^{-\frac{3}{2}} e_0 \sum_{k=1}^{n-2} \left(1 - \cos\frac{2\pi k}{n-1}\right)^{-\frac{1}{2}}. \tag{1.1}$$

From the Earnshaw theorem mentioned in [6] it follows that these equilibria are unstable. In [7] there is a profound sketch how to prove it.

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http://dx.doi.org/10.1016/j.geomphys.2015.08.012
0393-0440/

In general the existence of equilibrium configurations (they are absent in the celestial mechanics in fixed coordinate axes) may offer an application of the Lyapunov center theorem and as a result a construction of periodic solutions of corresponding equations of motion. This construction is produced by us in [8] for two and three equal negative charges at a line interacting with fixed two positive charges outside the line.

The presented here exact solutions are periodic functions in time and describe the motion of every negative charge as a keplerian motion with a fixed at the origin positive charge. Besides the negative charges are tied to vertices of regular polygons all the time. In the simplest case the negative charges rotate around the positive charge with the same frequency. This is described by the following solution $x_k(t) \in \mathbb{C}$ of the equation of motion for negative charges

$$x_k(t) = z_k e^{iu(t)} r(t), \quad r(t) > 0, \tag{1.2}$$

where $z_k, u(t), r(t)$ satisfy the first, second and third structure equations, respectively, the first of which is stationary and z_k does not depend on time. If one proves that $e_n^0 < (n-1)e_0$ then the found solution will establish the existence of a classical atom. We prove this for $n < 14$ in the Proposition 2.2.

We started this investigation trying to find the Lagrange-type triangular solutions of the Coulomb equation of motion for three planar charges. It is known that the Coulomb point charges e_j have the equilibrium configuration on a line such that the negative charges are located at the same distance from the positive charge if $e_1 = e_2 = -e_0, 4e_3 = e_0 > 0$. If $4e_3 > e_0$, the masses of the equal charges are also equal and they are at the same distance from the positive charge on a common line then it is natural to expect that they can rotate around the positive immobile charge in such the configuration (the centrifugal force will be equilibrated by the attraction). We confirm this conjecture in the Proposition 3.2 and show in the Proposition 3.3 that a rectilinear uniform rotation of the charges took place if $e_3 \geq -e_j, j = 1, 2$. In the Proposition 3.3 we prove that there are no triangular Lagrange-type of the uniformly rotating three charges. The ideas of the proofs are inspired by [3].

In the second section we explain in the Proposition 2.1 how to construct the exact solutions in Coulomb systems when negative charges are equal and there are equilibrium configurations in the form of regular polygons. Theorem 2.1 establishes their existence.

2. General dynamics

Let us consider the Coulomb system of n charges $e_j \in \mathbb{R}, j = 1, \ldots, n$ with masses $m_j > 0$. Their coordinates $x_j \in \mathbb{R}^d$ obey the Coulomb equation of motion

$$m_j \frac{d^2 x_j}{dt^2} = -\frac{\partial U(x_{(n)})}{\partial x_j}, \quad j = 1, \ldots, n,$$

where

$$x_{(n)} = (x_1, \ldots, x_n) \in \mathbb{R}^{dn}, \qquad U(x_{(n)}) = \sum_{1 \leq j < k \leq n} \frac{e_j e_k}{|x_j - x_k|}, \qquad |x|^2 = (x^1)^2 + \cdots + (x^d)^2.$$

Let $d = 2$ and $x_k = x_k^1 + ix_k^2 \in \mathbb{C}$. Then the Lagrange-type solutions are determined by the equalities $x_k = z_k q(t)$ which give

$$|x_j - x_k| = |z_j - z_k||q|.$$

From

$$m_j \frac{d^2 x_j}{dt^2} = \sum_{k=1, k \neq j}^{n} e_j e_k \frac{x_j - x_k}{|x_j - x_k|^3}, \quad j = 1, \ldots, n,$$

it follows that

$$m_j z_j \frac{d^2 q}{dt^2} = q|q|^{-3} \sum_{k=1, k \neq j}^{n} e_j e_k \frac{z_j - z_k}{|z_j - z_k|^3}.$$

Let

$$q(t) = e^{iu(t)} r(t), \quad r(t) > 0$$

and the following first structure equation hold

$$-w^2 m_j z_j = \sum_{k=1, k \neq j}^{n} e_j e_k \frac{z_j - z_k}{|z_j - z_k|^3}.$$

Then the equation of motion yields the following equation

$$e^{-iu(t)} \frac{d^2 q}{dt^2} + w^2 r^{-2} = \frac{d^2 r}{dt^2} + i\left(2\frac{dr}{dt}\frac{du}{dt} + \frac{d^2 u}{dt^2} r\right) - \left(\frac{du}{dt}\right)^2 r + w^2 r^{-2} = 0.$$