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A note on tame/compatible almost complex structures on four-dimensional Lie algebras ☆

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ABSTRACT

Four-dimensional, oriented Lie algebras $\mathfrak{g}$ which satisfy the tame–compatible question of Donaldson for all almost complex structures J on $\mathfrak{g}$ are completely described. As a consequence, examples are given of (non-unimodular) four-dimensional Lie algebras with almost complex structures which are tamed but not compatible with symplectic forms.

1. Introduction

Among other interesting problems on compact 4-manifolds raised in [1], Donaldson asked the following:

Question 1.1. *If J is an almost complex structure tamed by a symplectic form, is J also compatible with a symplectic form?*

Recall that an almost complex structure J is said to be *tamed* by a symplectic form ω (and such an ω is called J*-tamed*), if ω is positive on J-planes, i.e.

$$\omega(u, Ju) > 0, \quad \text{for all vectors } u \neq 0.$$

An almost complex structure J is said to be *compatible* with a symplectic form ω (and such an ω is called *compatible* with J, or J*-compatible*), if ω is J-tamed and J-invariant, i.e.

$$\omega(u, Ju) > 0 \text{ and } \omega(Jv, Jw) = \omega(v, w), \quad \text{for all vectors } u \neq 0, v, w.$$

Question 1.1 is still open for compact 4-manifolds, although important progress has been made by Taubes [2] who answered the question affirmatively for generic almost complex structures on 4-manifolds with $b^+ = 1$. There are other significant positive partial results, e.g. see [3,4], as well as results on the symplectic Calabi–Yau problem [5–8], also proposed by Donaldson in [1] and known to imply an affirmative answer to Question 1.1 for compact 4-manifolds with $b^+ = 1$. It is also worth noting that Donaldson's question is true locally for all almost complex 4-manifolds, but this is no longer the case in higher dimensions, as for certain J's the structure of their Nijenhuis tensors becomes a local obstruction to the existence of compatible symplectic forms (see e.g. [9–11]). There are no such obstructions for *integrable* almost complex structures

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and an important version of Donaldson question ([3], p. 678, and [12], Question 1.7) is whether it holds for compact complex manifolds of arbitrary dimensions. This is known to be true for compact complex surfaces [3]; for higher dimensions there are known only some partial results, e.g. [13–16].

Question 1.1 has an obvious Lie algebra version, which has been already considered. Indeed, on a Lie algebra $\mathfrak{g}$ an almost complex structure is an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ with $J^2 = -1$, and we talk about symplectic (or closed, or exact) forms on $\mathfrak{g}$ with respect to the Chevalley–Eilenberg differential d induced by the Lie bracket. Let us denote by $\mathcal{Z}^2$ the space of closed 2-forms and by B_2 the space of boundary 2-vectors on $\mathfrak{g}$. The space B_2 can be defined as the annihilator of $\mathcal{Z}^2$ with respect to the natural pairing between forms and vectors, that is $u \in B_2$ if and only if $\alpha(u) = 0$, for all $\alpha \in \mathcal{Z}^2$. For convenience, we introduce the following.

Definition 1.2. An oriented Lie algebra $\mathfrak{g}$ is said to satisfy the *tame–compatible property* if the answer to Question 1.1 is affirmative for **every** almost complex structure J on $\mathfrak{g}$ inducing the given orientation.

For the 4-dimensional Lie algebra version of Question 1.1, one main result of Li and Tomassini in [17] is the following:

Theorem (*[17], Theorem 0.2*)**.** *On a four-dimensional Lie algebra $\mathfrak{g}$, if the space of boundary 2-vectors B_2 is isotropic with respect to the wedge product, that is, if $u \wedge u = 0$, for all $u \in B_2$, then $\mathfrak{g}$ satisfies the tame–compatible property.*

As pointed out in [17], any four-dimensional unimodular Lie algebra satisfies the condition of the theorem, thus, satisfies the tame–compatible property. A consequence is that Question 1.1 has an affirmative answer for any left-invariant almost complex structure on a compact quotient of a 4-dimensional Lie group by a discrete subgroup (Theorem 4.3, [17]). It is well known that if a Lie group admits a co-compact discrete subgroup then its Lie algebra must be unimodular [18]. Note also that the assumption in the above result is independent of the choice of orientation on $\mathfrak{g}$, hence the conclusion is valid for both orientations. Although it covers the important unimodular case, the above result of Li and Tomassini gives only a sufficient condition for a 4-dimensional Lie algebra to satisfy the tame–compatible property. We show that a slightly weaker condition, which takes into account orientation, is, in fact, sufficient and necessary for the tame–compatible property.

Theorem 1.3. *Let $\mathfrak{g}$ be an oriented symplectic four-dimensional Lie algebra with a volume form μ. Then $\mathfrak{g}$ satisfies the tame–compatible property if and only if the space of boundary 2-vectors B_2 is negative semi-definite with respect to the bilinear form defined by the wedge product and the volume form, that is, if and only if $\mu(u \wedge u) \leq 0$, for all $u \in B_2$.*

The proof of the "if" direction could be obtained by refining the arguments of Li–Tomassini (with slight adjustments, a version of the Theorem 2.5 [17] still holds). However, partly to make our note self-contained and partly to present a different proof, in Section 3 we prefer to cast the 4-dimensional tame–compatible problem in an abstract linear algebra setting. We prove two linear algebra results (Propositions 3.2 and 3.3) which might have some independent interest. Theorem 1.3 follows directly from Proposition 3.2, as shown in Section 4.

Using the classification of four-dimensional symplectic Lie algebras obtained by Ovando [19], and her notations, there are two examples (or, rather, one and one-half!) for which B_2 is not negative semi-definite.

Corollary 1.4. *On the Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$ endowed with either orientation, or on the Lie algebra $\mathfrak{d}_{4,2}$ endowed with the non-complex orientation there exist almost complex structures which are tamed by symplectic forms but which are not compatible with any symplectic forms. These are the only 4-dimensional Lie algebras carrying such almost complex structures.*

In Section 4 we give the bracket descriptions of the two Lie algebras mentioned above. Although Corollary 1.4 follows directly from Theorem 1.3, we provide in each case explicit examples of almost complex structures which are tamed but not compatible. Let us just mention here that $\mathfrak{d}_{4,2}$ is the Lie algebra underlying the unique proper 4-dimensional example of 3-symmetric space discovered by Kowalski [20]. With one orientation this Lie algebra admits a complex (in fact, Kähler) structure, with the other orientation it does not admit complex structures. This is the orientation which we call "non-complex". Note also that $\mathfrak{d}_{4,2}$ does admit symplectic structures for both orientations.

Section 4 ends with two observations related to the main result of the paper. As a direct consequence of Lemma 3.2 in [13], Proposition 4.1 shows that any non-abelian even-dimensional Lie algebra admits almost complex structures which are not tamed by symplectic forms. Then in Proposition 4.2 we show that on any oriented 4-dimensional Lie algebra with $\dim(\mathcal{Z}^2) = 5$, every almost complex structure is either integrable or compatible with a symplectic form, but not both. This last result was prompted by a question of Tian-Jun Li and can be seen as an extension of Corollary 3.8 in [17].

To end the introduction, let us note that it was well known that the Lie algebra version of Question 1.1 can have negative answer for almost complex structures on Lie algebras of dimension 6 or higher, even in the unimodular case. The first such examples are due to Migliorini and Tomassini [9] (see also [10,21,22]). It would be interesting to know if Theorem 1.3 extends in any way for certain Lie algebras of dimensions higher than 4. Obviously, on an abelian Lie algebra of arbitrary even-dimension all almost complex structures are tamed and compatible with symplectic forms, but it would be interesting to find other classes of Lie algebras which satisfy the tame–compatible property and eventually classify them. We leave this open for future work.