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On pseudo-Finsler manifolds of scalar flag curvature

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ABSTRACT

Let (M, L) be a pseudo-Finsler manifold, ξ the geodesic spray vector field associated to the non-degenerate, 2-positively homogeneous Lagrangian L. In this paper we prove that (M, L) is of scalar flag curvature k if and only if the equation $\mathcal{L}_\xi g + k\lambda\mathcal{L}_\xi \hat{g} = 0$ holds on $\Gamma(I_\lambda M)$, the Lie algebra of tangent vector fields to the λ-indicatrix bundle $I_\lambda M$, where g and $\hat{g}$ are pseudo-Riemannian metrics on the vertical and respectively on the horizontal subbundle. Also, we prove that any pseudo-Finsler manifold is of scalar flag curvature at any point of the light cone.

1. Introduction

The theory of Finsler manifolds of scalar flag curvature, in particular of constant flag curvature, is an important subject in Finsler geometry which has been studied in numerous papers, many of them published in the last two decades. The excellent monograph of Bao–Chern–Shen [1] contains a detailed description of this topic and an extensive number of references related to it.

The relationship between a Riemannian manifold of constant curvature and the geometry of its tangent bundle was first investigated by Tashiro [2] and then by Tano [3] who proved that for a Riemannian manifold (M, g°) of constant curvature k, the geodesic spray ξ of the unit sphere bundle with the Sasaki metric is a Killing vector field if and only if $k = 1$. Their results were later extended to pseudo-Riemannian manifolds by Beem–Chicone–Ehrlich [4] and to Finsler manifolds by Bejancu–Farran [5].

In this paper we obtain a characterization of pseudo-Finsler manifolds of scalar flag curvature by proving that a pseudo-Finsler manifold (M, L) is of scalar flag curvature $k = k(y)$ if and only if the equation $\mathcal{L}_\xi g + k\lambda\mathcal{L}_\xi \hat{g} = 0$ holds on $\Gamma(I_\lambda M)$, the Lie algebra of tangent vector fields to the λ-indicatrix bundle $I_\lambda M$. Here ξ is the geodesic spray (the second order vector field) associated to the nondegenerate, 2-positively homogeneous Lagrangian L, the symmetric bilinear forms $g := \frac{1}{2}\omega \circ (id \times Q)$ and $\hat{g} := \frac{1}{2}\omega \circ (J \times id)$ are pseudo-Riemannian metrics on the vertical and respectively horizontal subbundles, J is the canonical tangent structure of the tangent bundle TM, $\omega = d(dL \circ J)$ is the canonical symplectic structure of (M, L) and $Q = J^{-1}_{|VTM^\circ}$.

An important step of our approach is the study of the scalar flag curvature on the light cone $I_0M := \{y \in TM^\circ \mid L(y) = 0\}$. We prove that any pseudo-Finsler manifold is of scalar flag curvature at any point of the light cone.

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Using the above characterization of pseudo-Finsler manifolds, we prove that if (M, L) is of scalar flag curvature $k = k(y)$, then the restriction of the geodesic spray ξ to the λ—indicatrix bundle $I_\lambda M$ is a Killing vector field with respect to the induced Sasaki metric if and only if (M, L) is of constant flag curvature $k = \frac{1}{\lambda}$. The main results in [3–5,2] are immediate consequences of our theorem.

Our results are related to pseudo-Finsler metrics of scalar flag curvature, i.e., the flag curvature being independent of flags but generally not of $y \in T_xM$. For Riemannian metrics the flag curvature is independent of $y \in T_xM$, however, Finsler metrics of scalar flag curvature are abundant. For example locally projectively flat Finsler metrics are of scalar flag curvature [6].

The technique that is used throughout the paper, apart that is coordinate free, is based on very known properties of some geometric structures associated to a specific Lagrangian. In this new approach the vertical subbundle has been used to incorporate few concepts of Finsler geometry in a clear geometric description, within the geometry of the tangent bundle. The scalar flag curvature is related to the vertical bundle and is defined as the sectional curvature of the vertical plane sections that contain the Liouville vector field, while the Riemannian curvature is a vertical vector bundle morphism.

2. Preliminaries

Let M be a connected, paracompact, smooth n-dimensional manifold, (TM, τ) its tangent bundle and $\Omega^k(M)$ the module of differentiable k-forms on M. The $C^\infty(M)$ — Lie algebra of vector fields on M is denoted by $\mathfrak{X}(M)$ and the $C^\infty(TM)$ — Lie algebra of vector fields on TM is denoted by $\mathfrak{X}(TM)$. The kernel of the tangent map τ_* is the *vertical subbundle*, $VTM := \ker \tau_*$. The sections of VTM, called *vertical vector fields*, form a $C^\infty(TM)$-Lie subalgebra of $\mathfrak{X}(TM)$, denoted by $\mathfrak{X}^v(TM)$.

A *vector k-form* on M is a vector field on M if $k = 0$ and a skew-symmetric bundle morphism $\Phi : (TM)^k \to TM$ if $k > 0$. The space of such forms is a $C^\infty(M)$—module and is denoted by $\Omega^k(M, TM)$. Notice that any $\Phi \in \Omega^k(M, TM)$ is a tensor of type $(1, k)$ on M and may be thought as a skew-symmetric k-multilinear map $(\mathfrak{X}(M))^k \to \mathfrak{X}(M)$.

The *Frölicher–Nijenhuis bracket* of a vector field $X \in \mathfrak{X}(M)$ and the vector 1-form $K \in \Omega^1(M, TM)$ is the vector 1-form

$$\mathcal{L}_X K = [X, K] := \mathcal{L}_X \circ K - K \circ \mathcal{L}_X, \tag{2.1}$$

where $\mathcal{L}_X$ stands for the Lee derivative operator with respect to X.

Notice that some times throughout the paper the notation "$\circ$" is used to emphasize the composition of functions but most of the time, when it is no danger of confusion, we just omitted it.

There are some canonical geometrical objects which are essential in the study of the geometry of the tangent bundle. Two of them are the following:

- The *vertical Liouville vector field*

$$C \in \mathfrak{X}^v(TM), \quad C(y) = \left.\frac{d}{dt}\right|_{t=0} (y + ty), \quad y \in TM.$$

- The *vertical endomorphism* (the *tangent structure*) $J \in \Omega^1(TM, TTM)$,

$$J_y(w) = \left.\frac{d}{dt}\right|_{t=0} \left(y + t\tau_{*y}(w)\right), \quad y \in TM, \ w \in T_yTM.$$

The following properties are immediate (see [7–9]):

$$J^2 = 0, \qquad \ker J = \operatorname{Im} J = VTM, \qquad \mathcal{L}_C J = -J. \tag{2.2}$$

The above third equation shows that J is 0-positively homogeneous.

Throughout this paper we also consider the slit tangent bundle (TM°, τ°) i.e., the tangent bundle with the zero section removed.

Definition 1. Let $L : TM \to \mathbb{R}$ be continuous and C^∞ on TM°. The pair (M, L) is called a pseudo-Finsler manifold if

(a) $L(\lambda y) = \lambda^2 L(y)$, $\lambda > 0$, $y \in T_xM$, $x \in M$,
(b) The canonical two form $\omega := d(dL \circ J)$, restricted to TM°, is nondegenerate.

Notice that

$$\omega \circ (J \times id) + \omega \circ (id \times J) = 0, \tag{2.3}$$

(see [7], page 301) hence the mapping

$$g : \mathfrak{X}^v(TM^\circ) \times \mathfrak{X}^v(TM^\circ) \to C^\infty(TM^\circ), \qquad g(X, Y) := \frac{1}{2}\omega(J\tilde{X}, \tilde{Y}), \tag{2.4}$$

with $J(\tilde{X}) = X$, $J(\tilde{Y}) = Y$, is a well defined, nondegenerate symmetric bilinear form, which generates a pseudo-Riemannian metric on the vertical bundle. This metric will also be called the *pseudo-Finsler metric* of (M, L).

If g is positive definite then (M, L) is called a *Finsler manifold*. In this case the *Finsler metric function* is $F = L^{1/2}$.