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Scalar curvature and projective compactness

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ABSTRACT

Consider a manifold with boundary, and such that the interior is equipped with a pseudo-Riemannian metric. We prove that, under mild asymptotic non-vanishing conditions on the scalar curvature, if the Levi-Civita connection of the interior does not extend to the boundary (because for example the interior is complete) whereas its projective structure does, then the metric is projectively compact of order 2; this order is a measure of volume growth towards infinity. This implies a host of results including that the metric satisfies asymptotic Einstein conditions, and induces a canonical conformal structure on the boundary. Underpinning this work is a new interpretation of scalar curvature in terms of projective geometry. This enables us to show that if the projective structure of a metric extends to the boundary then its scalar curvature also naturally and smoothly extends.

1. Introduction

Throughout this article we consider a smooth manifold $\overline{M}$ of dimension $n+1$ with boundary ∂M and interior M, and the basic topic is that of relating geometric structures on M to geometric structures (in general of a different type) on ∂M. Apart from their intrinsic interest in differential geometry, questions of this type play an important role in several other areas of mathematics (e.g. scattering theory) and theoretical physics (e.g. general relativity and the AdS/CFT-correspondence), see the introduction of [1] for a more detailed discussion.

In particular we are interested in a problem of the following nature. Suppose we start with a geometric structure on M, which on its own does not admit a smooth extension to $\overline{M}$, in such a way that the boundary ∂M is "at infinity" in a suitable sense. Then one may ask whether some weakening of the structure in question does admit such an extension, and whether this extension induces a structure on ∂M linked to the interior geometry. A classical example of this situation, with many applications, involves a notion of conformal extension. In this case, one starts with a pseudo-Riemannian metric g on M that does not admit a smooth extension to $\overline{M}$, for example because it is complete. Then one may first ask whether the conformal structure $[g]$ on M, determined by g, admits a smooth extension to all of $\overline{M}$. Explicitly, this means that, for each boundary point $x \in \partial M$, there is an open neighborhood $U \subset \overline{M}$ of x and a smooth nowhere vanishing function $f : U \cap M \to \mathbb{R}_{>0}$ such that the pseudo-Riemannian metric fg on $U \cap M$ admits a smooth extension to all of U for which the values on $U \cap \partial M$ are non-degenerate as bilinear forms on tangent spaces.

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In most applications, this idea is refined to the more restrictive, but also more useful, concept of *conformal compactness*. Rather than assuming an arbitrary smooth rescaling of g extends to the boundary, one requires that r^2g admits a smooth extension to the boundary, where $r : U \to \mathbb{R}_{\geq 0}$ is a *local defining function* for the boundary, see Section 2.1 for the formal definition. This enforces a certain uniformity in the growth rate of the metric g as it approaches the boundary. The property that r^2g admits a smooth extension to the boundary is independent of the specific defining function r, and it follows that the conformal class of the induced pseudo-Riemannian metric on ∂M is also independent of r. This conformal class is then the induced structure on the boundary, and the boundary so equipped is then referred to as the *conformal infinity* of the interior.

Alternative to the underlying conformal structure of a pseudo-Riemannian metric g, one may also consider the underlying projective structure of its Levi-Civita connection ∇^g. The resulting applications of projective differential geometry to pseudo-Riemannian geometry have been intensively and successfully studied during the last years. Because of the resulting emphasis on geodesic paths, this approach should be particularly useful for applications in general relativity and scattering theory. Indeed, as brought to our attention by P. Nurowski, there have been attempts to associate a future projective infinity to space–times, see [2].

In the setting of a manifold with boundary $\overline{M} = M \cup \partial M$ as above, one can start from a torsion free linear connection on TM, which does not extend to $\overline{M}$ and assume that its underlying projective structure extends to $\overline{M}$. Via the Levi-Civita connection, this concept is then automatically defined for pseudo-Riemannian metrics. Our first result in this article is an explicit characterization of extendability of the projective structure in Proposition 2.

In analogy with the way in which conformal compactification usefully restricts conformal extension (as discussed above) a concept of *projective compactness*, describing a special type of projective extension, was introduced in our articles [1,3] and studied further in [4]. This involves a parameter $\alpha > 0$, called the order of projective compactness, and one usually assumes that $\alpha \leq 2$. The latter ensures that ∂M is at infinity according to the parameters of geodesics approaching the boundary, see Proposition 2.4 in [1]. For a projectively compact connection that preserves a volume density, α is a measure of growth rate of this volume density towards the boundary at infinity, see [1] and Section 2.1.

There are two results in [1] which motivate the developments in this article. Assume that ∇ is a linear connection on TM, which does not admit a smooth extension to any neighborhood of a boundary point, but whose projective structure does admit a smooth extension to all of $\overline{M}$. Then in Theorem 3.3 of [1] it is shown that if ∇ preserves a volume density and is Ricci flat, then it is projectively compact or order $\alpha = 1$. On the other hand, if ∇ is the Levi-Civita connection of a non-Ricci-flat Einstein metric, then Theorem 3.5 of [1] shows that ∇ is projectively compact of order $\alpha = 2$. In both cases, one actually obtains reductions of projective holonomy, which lead to much more specific information.

The main result of this article is Theorem 5, which provides a vast generalization of the second of these results. We replace being Einstein by a much weaker condition on the asymptotics of the scalar curvature of g, but we still can conclude projective compactness of order $\alpha = 2$. Via the results of [4], this provides a number of further facts about g, including a certain asymptotic form, an asymptotic version of the Einstein property, and the fact that ∂M inherits a canonical conformal structure determined by g. Some of these consequences are summarized in Corollary 6.

The results in Theorem 5, along with converse results in [4], expose a previously unseen critical role for the scalar curvature in questions of projective compactification. In Proposition 3 we show that if the projective structure of a metric on M extends to $\overline{M}$ then, surprisingly, its scalar curvature also extends smoothly (as a function) to $\overline{M}$. The way this works is also important for our treatment. Several of the arguments used in proving this result should be of considerable independent interest.

2. Results

2.1. Projective structures and projective compactness

Two torsion free linear connections ∇ and $\hat{\nabla}$ on the tangent bundle of a smooth manifold N are called *projectively equivalent* if they have the same geodesics up to parametrization or, equivalently, if there is a one-form $\Upsilon \in \Omega^1(N)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi, \tag{1}$$

for vector fields $\xi, \eta \in \mathfrak{X}(N)$. A *projective structure* on N is a projective equivalence class of such connections.

Assume that $\overline{M}$ is a smooth manifold with boundary ∂M and interior M. Given a torsion free linear connection ∇ on TM, we can define what it means for the projective structure determined by ∇ to admit a smooth extension to all of $\overline{M}$. Explicitly, this is the property that, for any boundary point $x \in \partial M$, there is an open neighborhood U of x in $\overline{M}$ and a one-form $\Upsilon \in \Omega^1(U \cap M)$ such that for all vector fields $\xi, \eta \in \mathfrak{X}(U)$ (so these are smooth up to the boundary), also $\hat{\nabla}_\xi \eta$ as defined in (1) admits a smooth extension to the boundary. It is then clear that the resulting connection $\hat{\nabla}$ on TU is uniquely determined up to projective equivalence, so in this way one indeed obtains an extension of the projective structure to $\overline{M}$.

More restrictively, for a constant $\alpha > 0$ we say that ∇ is *projectively compact of order* α, if in the above considerations the one-form Υ can be taken to be $\frac{dr}{\alpha r}$ for a smooth defining function $r : U \to \mathbb{R}_{\geq 0}$ for the boundary. By definition, the latter condition means that $r^{-1}(\{0\}) = U \cap \partial M$ and dr is nowhere vanishing on $U \cap \partial M$.